Time Delayed Feedback From Active Media

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Bachelor of Science

in Physics

Statement of Authorship

This thesis has been submitted for the degree of Bachelor of Science in Physics. I, the undersigned, hereby declare that:

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Fabian Faulstich

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Abstract

We first consider a dielectric medium of identical two-state atoms coupled by the radiation field to an initially excited atom outside the dielectric medium. By using a unitary operator the describing Hamiltonian is transformed into the rotating frame with respect to the transition frequency of the atom ω_0. Schrödingers equation now yields a system of differential equations describing the interaction of the atom with the dielectric medium vie virtual photon exchanges. Assuming that the atoms in the dielectric medium are continuously distributed, we derive an alternative delay differential equation in which a Fresnel reflection coefficient appears. We further calculate an analytical solution of this delay differential equation to an initial value. Analogously to Schrödingers equation Heisenbergs equation yields a similar system of differential operator equations describing the interaction of the atom with the dielectric medium vie virtual photon exchanges. In the macroscopic limit of a continuous distribution of atoms in the dielectric medium, we derive a delay differential operator equation. This equation is the Heisenberg picture analogon to the before derived delay differential equation. Further a numerical simulation of the initially excited atoms excited state is presented.

Zusammenfassung

Wir betrachten zunächst ein Dielektrikum bestehend aus Zwei-Niveau Atomen, welches über ein Strahlungsfeld an ein anfänglich angeregtes Atom koppelt. Unter Verwendung eines unitären Operators transformieren wir den beschreibenden Hamiltonian in den rotierenden Rahmen bzgl. der Übergangsfrequenz des anfänglich angeregten Atoms ω_0. Ausgehend von der Schrödingergleichung leiten wir ein System von Differentialgleichungen her, das die Wechselwirkung des Atoms mit dem Dielektrikum via virtuellem Photonenaustausch beschreibt. Unter der Annahme, dass die Atome im Dielektrikum kontinuierlich verteilt sind leiten wir eine alternative zeitverzögerte Differentialgleichung mit einem Fresnel-Reflexionskoeffizient her. Des Weiteren bestimmen wir eine analytische Lösung der zeitverzögerten Differentialgleichung zu einem gegebenen Anfangswert. Analog zur Schrödingergleichung folgt aus der Heisenberg'schen Bewegungsgleichung ein ähnliches System von Operator Differentialgleichungen, das die Wechselwirkung des Atoms mit dem Dielektrikum via virtuellem Photonenaustausch beschreibt. Im makroskopischen Limes eines kontinuierlich verteilten Dielektrikums leiten wir eine zeitverzögerte Operator Differentialgleichung her. Diese Gleichung ist das Analogon der zeitverzögerten Differentialgleichung des Schrödinger Bildes. Abschließend ist eine numerische Simulation des angeregten Zustandes, des zu beginn angeregten Atoms, vorgeführt.

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Contents

1 Introduction

In 1927 a collection of information was published by Niels Bohr and Werner Heisenberg that postulated how the mathematical formalism of quantum mechanics is supposed to be understood in terms of everyday language [1]. This publication gave a unique interpretation of the analysed quantities. This so-called *Copenhagen interpretation* was the start of a period full of new theories and models which aimed to describe physical systems. Since then there has been huge progress in the field of theoretical physics, especially in quantum theory.

Quantum cavity electrodynamics is one of the fields that has come up since then. Inter alia it aims to describe the control of quantum systems. As real systems are never perfectly isolated they always interact with their environment. This interaction can not only be so strong that it is not negligible but so strong that it highly affects the behaviour of the system. This is called *control* of the system. Therefore the analysis of such systems is of interest, not only for theoretical physicists but for all fields of study that analyse any sort of quantum information processing network. This thesis will focus on the control through feedback from the environment. It will also lay the foundations for the analysis of feedback trough excited quantum media which is the control through input or *gain*.

We will start with the Hamiltonian describing an atom interacting with a dielectric medium by virtual photon exchanges. After a transformation into the rotating frame with respect to the transition frequency of the initially excited atom we will derive a system of differential equations describing the probability amplitudes of the state function in Schrödingers picture. These differential equations are the fundamentals of *'Quantum theory of an atom near partially reflecting walls'* by R.J. Cook and P. W. Milonni [2]. We will then give a rigorous derivation of the refraction coefficient vie a frequency depending coupling to the quantised electromagnetic field. This will reduce the system of differential equations to one delay differential equation describing the initially excited atom. This delay differential equation will have only an initial value. The analysis of this equation will yield that the initial value is enough to calculate a unique solution. The existing solution is then calculated and proven. After the analysis of the problem using Schrödingers picture, we will transfer into Heisenbergs picture. Here the system of describing differential equations will be given by suitable operator equations. The initial values of the system describing differential equations are mostly given by the physical properties of the system. As the physical interpretation of operators is unclear, only the expected values can be identified with physical properties of the system. This in combination with the commutator relations of the operators leads to *quantum noise*. The calculations used in Schrödingers picture are mostly transferable and will lead to a delay differential operator equation. Due to the one excitation limit, normal ordering is prevented and an initially empty vacuum is assumed. This will allow us to omit the quantum noise term and we will obtain an operator analogon to the equation in Schrödingers picture. This equation will be dependent of the operator describing the excited state of the initially excited atom. Finally we will analyse this operator and present a solution with a numerical simulation.

2 Formulations of Quantum Mechanics

In this section we will introduce the quantum mechanical framework this thesis will make use of. We start with a mathematical background of the Hamilton operator which will lead to the definition of the *Sobolev space* H^1. We will further demonstrate the *weak formulation* of Schrödingers equation and define the domain of an N-particle Hamiltonian of a fermionic system. Further we will describe *Schrödingers* and *Heisenbergs picture*. Both are formulations of quantum mechanics but they differ in the description of the systems time dynamics. We emphasise that only the prerequisites necessary for this thesis are mentioned. For more detailed information see [3], [4], [5], [6] and [7].

2.1 Hamiltonoperator

The Hamilton operator will be the main object of this thesis. Therefore we will shortly introduce this differential operator and its domain. Due to simplicity we may only describe static, real valued problems. The following construction is geared to [3]. For more detailed information see [4], [5] and [6].
To define the necessary functional spaces we will use the *multi-index notation*.

Definition 2.1. *An n-dimensional multi-index is an n-tuple*

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \tag{2.1}$$

of non-negative integers (i.e. an element of the n-dimensional set of natural numbers, denoted $\mathbb{N}_0^n$). For the multi-index $\alpha \in \mathbb{N}_0^n$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and the $|\alpha|$-times continuously differential function $f : \mathbb{R}^n \to \mathbb{C}$ we define:

(i) the sum of components (absolute value) of a multi-index by

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n \tag{2.2}$$

(ii) the multi-index power of a vector by

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \tag{2.3}$$

(iii) the multi-index derivative by

$$D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} f \ . \tag{2.4}$$

Using this notation we can define the *space of rapidly decreasing functions on* $\mathbb{R}^n$. This space also known as *Schwartz space* plays an important role in Fourier analysis and provides an approach to the *Sobolev spaces.*

Definition 2.2. *The **Schwartz space** or **space of rapidly decreasing functions on** $\mathbb{R}^n$ is the function space*

$$\mathcal{S}\left(\mathbb{R}^n\right) = \left\{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty \quad \forall \alpha, \beta\right\}, \tag{2.5}$$

where α, β are multi-indices, $\mathcal{C}^\infty(\mathbb{R}^n)$ is the set of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$, and

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right| . \tag{2.6}$$

The *Sobolev space* H^1 and therewith the domain of the Hamiltonian is a closure of the Schwartz space under a specific norm. We emphasise that we are considering infinite dimensional functional spaces which implies that norms do not have to be equivalent. Hence, the closure of sets differs with the considered norm. The norm which yields the Sobolev space is the H^1*-norm*.

Definition 2.3. *The map*

$$||\cdot||_1^2 : L^2 \cap \mathcal{C}^1 \to L^2; u \mapsto ||u||_1^2 = ||u||_{L^2}^2 + ||\nabla u||_{L^2}^2 \ , \tag{2.7}$$

*is called H^1**-norm**. Here $||\cdot||_{L^2}$ is the norm induced by the L^2 scalar product $\langle\cdot,\cdot\rangle_{L^2}$ and $\mathcal{C}^1$ denotes the space of continuously differentiable functions.*

Using the above defined structures we define the *Sobolev space* H^1 as follows.

Definition 2.4. *The closure of the Schwartz space under the H^1-norm is called **Sobolev space** or H^1*

$$H^1(\mathbb{R}^n) = \overline{\mathcal{S}(\mathbb{R}^n)}^{\,||\cdot||_1} . \tag{2.8}$$

As mentioned before the Hamilton operator is a differential operator. As we say the Hamilton operator we always refer to a specific Hamilton operator namely the one considered in this thesis. In general a Hamilton operator is defined as follows.

Definition 2.5. *A **Hamilton operator** is an elliptical differential operator, defined by*

$$\mathcal{H}u = -\frac{1}{2}\Delta u + Vu \ . \tag{2.9}$$

*The function $V : \mathbb{R}^n \to \mathbb{R}$ is called the **potential** of the operator.*

As we can see a Hamilton operator contains the Laplace operator. Hence the domain of this operator can not be larger than the space of the twice derivable functions. As the ground state energy of a stationary system is the eigenvalue of the Hamilton operator Schrödingers equation is one of the elementary equations in quantum mechanics. We will therefore derive the weak formulation of Schrödingers equation which will be well defined on a bigger space than the Hamilton operator given in the above formulation.

We define $\mathscr{D} \subseteq \mathcal{S}$ consisting out of the functions that vanish outside bounded sets. Be $\varphi \in \mathscr{D}$ then Schrödingers equation yields

$$\begin{aligned} & E\psi = \mathcal{H}\psi \\ \Leftrightarrow \quad & \varphi E\psi = \varphi\mathcal{H}\psi \\ \Leftrightarrow \quad & \int \varphi E\psi = \int \varphi\mathcal{H}\psi = -\frac{1}{2}\int \varphi\Delta\psi + \int \varphi V\psi = \frac{1}{2}\int \nabla\varphi\nabla\psi + \int \varphi V\psi \ . \end{aligned} \tag{2.10}$$

Using the L^2 scalar product Schrödingers equation is equivalent to

$$\langle \varphi, E\psi\rangle_{L^2} = \frac{1}{2}\langle \nabla\varphi, \nabla\psi\rangle_{L^2} + \langle \varphi, V\psi\rangle_{L^2} \quad , \forall\varphi \in \mathscr{D} \ . \tag{2.11}$$

This holds true for the closure of $\mathscr{D}$ under the H^1-norm which again is H^1. This shows that a stationary Hamilton operator can be defined in a *weaker* form which yields a bigger space of solutions. This formulation of Schrödingers equation is called the *weak formulation of Schrödingers equation.* The operators used in the inner product are well defined on the above derived H^1. For an N-particle static fermionic problem the weak formulation of Schrödingers equation is well defined on

$$\mathscr{H} = H^1(\mathbb{R}^3 \times \{\pm\frac{1}{2}\})^N \cap \bigwedge_{i=1}^{N} L^2(\mathbb{R}^3 \times \{\pm\frac{1}{2}\}), \tag{2.12}$$

where $\wedge$ describes the antisymmetric tensor product of spaces which guarantees Paulis principle.
For an analogous construction of a non static problem see [3].

2.2 Schrödingers Picture

Schrödingers picture is a formulation of quantum mechanics in which state functions evolve in time but the operators do not. The following introduction to Schrödingers picture is geared to [7].
We consider a state $|\Psi_S(t=0)\rangle =: |\Psi_S\rangle_0$ described by initial conditions at the time $t = 0$. Due to special properties of the describing Hamiltonian $\mathcal{H}$ the solution to any time $t \in [0, T]$ is guaranteed. This solution is described by a so called *propagator*

$$U : [0, T] \times [0, T] \to L(\mathscr{H}); (t, s) \mapsto U(t, s) \ , \tag{2.13}$$

where $L(\mathscr{H}) = L(\mathscr{H}, \mathscr{H})$ is the space of linear and bounded operators on the considered Fockspace $\mathscr{H}$. The solution of Schrödingers equation is then given by

$$|\Psi_S(t)\rangle = U(t, 0)|\Psi_S\rangle_0 \ . \tag{2.14}$$

The state functions define probability density functions as

$$1 = \int_{\mathbb{R}^3} |\Psi_S(t)|^2 = ||\Psi_S(t)||_2^2 \qquad , \forall t \in [0, T] \tag{2.15}$$

holds true. This implies that the L^2-norm of state functions has to be constant in time. Hence,

$$\begin{aligned} & ||\Psi_S(0)||_2^2 = 1 = ||\Psi_S(t)||_2^2 = ||U(t,0)\Psi_S(0)||_2^2 \\ \Leftrightarrow \quad & U^\dagger(t,0)U(t,0) = Id \\ \Leftrightarrow \quad & U^\dagger(t,0) = U^{-1}(t,0) \ . \end{aligned} \tag{2.16}$$

In conclusion $U(t,0)$ is a unitary operator and therefore lengths and angles between elements in $\mathscr{H}$ are invariant under this operator. In Schrödingers picture the systems time dynamics is contained in the state functions. The operators $\mathscr{O}_S \in L(\mathscr{H})$ used to define the Hamiltonian are time independent but for their explicit time dependences

$$\frac{d}{dt}\mathscr{O}_S = \frac{\partial}{\partial_t}\mathscr{O}_S \ . \tag{2.17}$$

The time dynamic of state functions is described by the time dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi_S(t)\rangle = H_S|\Psi_S(t)\rangle \ . \tag{2.18}$$

2.3 Heisenbergs Picture

In Heisenbergs picture the operators evolve in time, but state functions do not. The following introduction to Heisenbergs picture is geared to [7].
One of the advantages of time evolving operators is that the number of excitations can differ and the calculated results still hold true. This is not the case for Schrödingers picture as here a change of the set of basis functions leads to complete different calculations. The state functions are now time independent, but coincide with the state functions in Schrödingers picture at the initial time $t = 0$

$$|\Psi_H(t=0)\rangle = |\Psi_H\rangle = |\Psi_S\rangle_0 \ . \tag{2.19}$$

As we consider stable physical systems, a solution of the Hamiltonian is still given by the nature of the considered problem. Hence there exists a propagator describing the solution denoted by $U(t,0)$. This yields

$$U^\dagger(t,0)|\Psi_S(t)\rangle = \underbrace{U^\dagger(t,0)U(t,0)}_{=Id}|\Psi_S\rangle_0 = |\Psi_H\rangle \ . \tag{2.20}$$

The transition into Heisenbergs picture is hence a unitary transformation and therefore the operators are transformed by

$$\mathscr{O}_H(t) = U^\dagger(t,0)\mathscr{O}_S U(t,0) \ . \tag{2.21}$$

As

$$\begin{aligned} \langle\mathscr{O}_S\rangle &= \langle\Psi_S(t)|\mathscr{O}_S|\Psi_S(t)\rangle = \langle\Psi_H|U^\dagger(t,0)\mathscr{O}_S U(t,0)|\Psi_H\rangle = \langle\Psi_H|\mathscr{O}_H(t)|\Psi_H\rangle \\ &= \langle\mathscr{O}_H(t)\rangle \end{aligned} \tag{2.22}$$

holds true, the expected values stay invariant under such unitary transformations. Considering the representation of the Hamiltonian in Schrödingers picture Schrödingers equation yields

$$i\hbar\frac{\partial}{\partial t}U(t,0)|\Psi_S\rangle_0 = H_S U(t,0)|\Psi_S\rangle_0 \quad \Leftrightarrow \quad i\hbar\frac{\partial}{\partial t}U(t,0) = H_S U(t,0)\ . \tag{2.23}$$

We will now analyse the total differential of the operator $\mathscr{O}_H(t)$

$$\frac{d}{dt}\mathscr{O}_H(t) = \left(\frac{\partial}{\partial t}U^\dagger(t,0)\right)\mathscr{O}_S U(t,0) + U^\dagger(t,0)\mathscr{O}_S\left(\frac{\partial}{\partial t}U(t,0)\right) + U(^\dagger t,0)\left(\frac{\partial}{\partial t}\mathscr{O}_S\right)U(t,0)\ . \tag{2.24}$$

Using (2.23) this yields

$$\begin{aligned}
\frac{d}{dt}\mathscr{O}_H(t) &= \left(\frac{i}{\hbar}U^\dagger(t,0)H_S\right)\mathscr{O}_S U(t,0) + U^\dagger(t,0)\mathscr{O}_S\left(-\frac{i}{\hbar}H_S U(t,0)\right) \\
&\quad + U(^\dagger t,0)\left(\frac{\partial}{\partial t}\mathscr{O}_S\right)U(t,0) \\
&= \frac{i}{\hbar}\Big(U^\dagger(t,0)H_S\mathscr{O}_S U(t,0) - U^\dagger(t,0)\mathscr{O}_S H_S U(t,0)\Big) + \left(\frac{\partial}{\partial t}\mathscr{O}_S\right)_H \\
&= \frac{i}{\hbar}\Bigg(\underbrace{U^\dagger(t,0)H_S U(t,0)}_{H_H(t)}\underbrace{U^\dagger(t,0)\mathscr{O}_S U(t,0)}_{\mathscr{O}_H(t)} \\
&\quad - \underbrace{U^\dagger(t,0)\mathscr{O}_S U(t,0)}_{\mathscr{O}_H(t)}\underbrace{U^\dagger(t,0)H_S U(t,0)}_{H_H(t)}\Bigg) + \left(\frac{\partial}{\partial t}\mathscr{O}_S\right)_H \\
&= \frac{i}{\hbar}\left[H_H(t),\mathscr{O}_H(t)\right] + \left(\frac{\partial}{\partial t}\mathscr{O}_S\right)_H\ ,
\end{aligned} \tag{2.25}$$

which is known as *Heisenbergs equation* . In (2.25) $[\cdot,\cdot]$ denotes the *Lie bracket.*
We have now set the mathematical frame for this thesis with two formulations of quantum mechanics and their elementary equations.

3 Description of Atom-Field Interaction Using Schrödingers Picture

In this section, we will analyse the interaction of a two state atom with an electromagnetic field and a dielectric medium consisting out of two state atoms. As the dynamics of the system in Schrödingers picture is given by the probability amplitudes of the state function space basis functions, we start by deriving the differential equations describing these. We will see that an orthogonal transformation of the Hamiltonian simplifies the calculations and leads to a much more compact representation of the describing differential equations. To derive this result the *Baker-Campbell-Hausdorff expanansion* will be the main tool.

The system of differential equations obtained in the first subsection is analytically not solvable. We will use physical properties of the considered system to simplify the differential equations. This will yield a solvable initial time problem with a delay differential equation describing the initially excited atom. We emphasise that for the unique solution of the time delay differential equation only an initial value will be needed and not the behaviour of the solution on an interval as usual for delay differential equations.

3.1 Derivation of Differential Equations Describing Probability Amplitudes

In the following we will discuss the interaction of a two-state atom with an electromagnetic field and a dielectric medium of N two-state atoms.

The position of the initially excited atom is denoted by x_0. We further denote its ground state with $|1\rangle$ and its excited state with $|2\rangle$. The state in which all atoms in the dielectric medium are in their ground state will be denoted by $|\{1...1\}\rangle$ and the state in which only the $j-th$ atom of the dielectric medium is excited will be denoted by $|\{1...\underbrace{1\,2\,1}_{\text{j-th atom}}...1\}\rangle$.

The considered electromagnetic field can contain one photon of mode $n \in \mathbb{N}$. The probability amplitude describing the state in which all atoms but the atom at x_0 are in their ground state and the field is in its vacuum state is given by $b(t)$. Equivalently the probability amplitude of the state in which the j-th atom in the dielectric medium is excited is given by $b_j(t)$. The map a_n denotes the probability amplitude for the state in which all the atoms are in their ground state and the field contains one photon of the n-th mode. This situation is illustrated in figure 1.

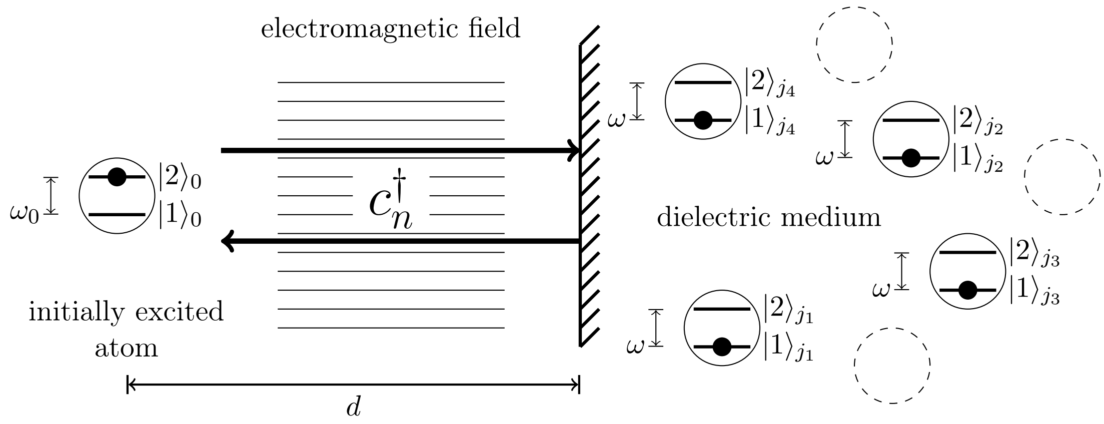

Figure 1: Excited atom located in a distance d to the dielectric medium.

Hence the describing state function is given by

$$|\psi\rangle = b(t)|2,0,\{1...1\}\rangle + \sum_{j=1}^{N} b_j(t)|1,0,\{1...\underbrace{1\ 2\ 1}_{\text{j-th atom}}...1\}\rangle + \sum_{n\in\mathbb{N}} a_n(t)|1,n,\{1...1\}\rangle\ . \quad (3.1)$$

In the following we will consider the so-called *flip-operators* $\sigma_{i,j}$ which are defined by

$$\sigma_{i,j} = |i\rangle\langle j|\ . \quad (3.2)$$

These operators change the state of an atom from $|j\rangle$ to $|i\rangle$. The atom-field interaction is described by the operator H_I which is given by

$$H_I = i\hbar \sum_{j=0}^{N} \sum_{n\in\mathbb{N}} C_n \left(\sigma_{2,1}^{(j)} c_n e^{ik_n\cdot x_j} - \sigma_{1,2}^{(j)} c_n^\dagger e^{-ik_n\cdot x_j} \right)\ . \quad (3.3)$$

The Hamiltonian for the system without interaction is given by

$$H_0 = \sum_{j=0}^{N} \hbar\omega_j \sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)} + \sum_{n\in\mathbb{N}} \hbar\omega_n c_n^\dagger c_n\ . \quad (3.4)$$

We will later assumed that the atoms in the dielectric medium all have the same transition frequency $\omega = \omega_1 = ... = \omega_N$. The aim of this section is to describe the initially excited atom at x_0 with one non coupled differential equation. We will therefore transfer into the rotating frame with respect to ω_0. This is an orthogonal transformation. We define the transformation Hamiltonian by

$$H_T = \hbar\omega_0 \sum_{j=0}^{N} \sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)} + \hbar\omega_0 \sum_{n\in\mathbb{N}} c_n^\dagger c_n \quad (3.5)$$

which is used to define the transformation

$$T : \mathscr{H} \to \mathscr{H}; f \mapsto exp\left(\frac{i}{\hbar} H_T t\right) f, \tag{3.6}$$

where $\mathscr{H}$ is the Fockspace of the above described system. As

$$T^* = \left(\sum_{m=0}^{\infty} \frac{(i/\hbar H_T t)^m}{m!}\right)^* = \sum_{m=0}^{\infty} \frac{((i/\hbar H_T t)^*)^m}{m!} = \sum_{m=0}^{\infty} \frac{(-i/\hbar H_T^* t)^m}{m!} = \sum_{m=0}^{\infty} \frac{(-i/\hbar H_T t)^m}{m!} \tag{3.7}$$

and $[H_T, H_T] = 0$ we conclude that

$$exp\left(\frac{i}{\hbar} H_T t\right) exp\left(-\frac{i}{\hbar} H_T t\right) = exp\left(\frac{i}{\hbar}(H_T - H_T)t\right) = Id\ . \tag{3.8}$$

Hence, T is indeed a unitary operator and describes an orthogonal transformation. The operator $\sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)}$ with $j \in \{0, ..., N\}$ acts on the j-th atom and the operator $c_n^\dagger c_n$ with $n \in \mathbb{N}$ acts on the photon of the mode n. They therefore act on different subspaces of the considered Fockspace $\mathscr{H}$ which are Hilbert spaces as they are complete subspaces of a Hilbert space. Hence $[\sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)}, c_n^\dagger c_n] = 0$ for all $j \in \{0, ..., N\}$ and $n \in \mathbb{N}$. Therefore we can express the operator T as

$$T = exp\left(i\omega_0 \sum_{j=0}^{N} \sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)} t\right) exp\left(i\omega_0 \sum_{n\in\mathbb{N}} c_n^\dagger c_n t\right)\ . \tag{3.9}$$

We will now analyse the orthogonal transformation of the atom-field interaction. Therefore we will use the *Baker-Campbell-Hausdorff expansion* that states:

Theorem 3.1. *For a linear and continuous operator X from a Banach space into itself the transformation $Z = e^X Y e^{-X}$ of the linear operator Y defined on the same Banach space into itself is given by*

$$Z = e^X Y e^{-X} = \sum_{m=0}^{\infty} \frac{1}{m!} [X, Y]_m\ , \tag{3.10}$$

where $[X, Y]_m = [X, [X, Y]_{m-1}]$ and $[X, Y]_0 = Y$. [7]

We start by analysing the term

$$\begin{aligned} &exp\left(i\omega_0 \sum_{j=0}^{N} \sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)} t\right) \sigma_{2,1}^{(J)} exp\left(-i\omega_0 \sum_{j=0}^{N} \sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)} t\right) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \left[i\omega_0 \left(\sum_{j=0}^{N} \sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)}\right) t\ ,\ \sigma_{1,2}^{(J)} \right]_m\ . \end{aligned} \tag{3.11}$$

We will now focus on the commutator relation in (3.11). We see that $[\cdot,\cdot]_1 = [\cdot,\cdot]$ and therefore analyse the term

$$\begin{aligned}\left[\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\ ,\ \sigma_{2,1}^{(J)}\right] &= \sum_{j=0}^{N}\left[\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\ ,\ \sigma_{2,1}^{(J)}\right] = \sum_{j=0}^{N}\delta_{j,J}\left[\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\ ,\ \sigma_{2,1}^{(J)}\right] \\ &= \sigma_{2,1}^{(J)}\left[\sigma_{1,2}^{(J)}\ ,\ \sigma_{2,1}^{(J)}\right] = \sigma_{2,1}^{(J)}\left(\sigma_{1,1}^{(J)} - \sigma_{2,2}^{(J)}\right) = \sigma_{2,1}^{(J)} - \sigma_{2,1}^{(J)}\sigma_{2,2}^{(J)}\ .\end{aligned} \tag{3.12}$$

By definition of the flip operator we obtain

$$\sigma_{2,1}^{(J)}\sigma_{2,2}^{(J)} = |2\rangle\langle 1|2\rangle\langle 2| = 0 \tag{3.13}$$

as $\langle i|j\rangle = \delta_{i,j}$. Hence

$$\left[\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\ ,\ \sigma_{2,1}^{(J)}\right] = \sigma_{2,1}^{(J)} \tag{3.14}$$

which yields

$$\left[\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\ ,\ \sigma_{1,2}^{(J)}\right]_m = \sigma_{2,1}^{(J)}\ . \tag{3.15}$$

Using this conclusion and the linearity of the Lie bracket we can rewrite term (3.11) as

$$\begin{aligned}&exp\left(i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t\right)\sigma_{2,1}^{(J)}exp\left(-i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t\right) \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\left[i\omega_0\left(\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\right)t\ ,\ \sigma_{2,1}^{(J)}\right]_m = \sum_{m=0}^{\infty}\frac{(i\omega_0 t)^m}{m!}\left[\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\ ,\ \sigma_{2,1}^{(J)}\right]_m \\ &= \sum_{m=0}^{\infty}\frac{(i\omega_0 t)^m}{m!}\sigma_{2,1}^{(J)} = e^{i\omega_0 t}\sigma_{2,1}^{(J)}\ .\end{aligned} \tag{3.16}$$

Analogously we find that

$$exp\left(i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t\right)\sigma_{1,2}^{(J)}exp\left(-i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t\right) = e^{-i\omega_0 t}\sigma_{1,2}^{(J)}\ , \tag{3.17}$$

$$exp\left(i\omega_0\sum_{n\in\mathbb{N}}c_n^\dagger c_n t\right)c_l exp\left(-i\omega_0\sum_{n\in\mathbb{N}}c_n^\dagger c_n t\right) = e^{-i\omega_l t}c_l\ , \tag{3.18}$$

and

$$exp\left(i\omega_0\sum_{n\in\mathbb{N}}c_n^\dagger c_n t\right)c_l^\dagger exp\left(-i\omega_0\sum_{n\in\mathbb{N}}c_n^\dagger c_n t\right) = e^{i\omega_l t}c_l^\dagger \tag{3.19}$$

holds true. For more detailed calculations see section (6.1.1). Applying these results to the interaction Hamiltonian yields

$$TH_IT^* = i\hbar \sum_{J=0}^{N} \sum_{l\in\mathbb{N}} C_l \left(e^{ik_l\cdot x_J} \sigma_{2,1}^{(J)} c_l - e^{-ik_l\cdot x_J} \sigma_{1,2}^{(J)} c_l^\dagger \right) \ . \tag{3.20}$$

A detailed calculation is presented in section (6.1.2). For the transformation of the Hamiltonian H_0 we remark that

$$\left[\sigma_{2,1}^{(J)}\sigma_{1,2}^{(J)} \ , \ \sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\right] = \left[c_l^\dagger c_l \ , \ c_n^\dagger c_n\right] = \left[\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)} \ , \ c_n^\dagger c_n\right] = 0 \tag{3.21}$$

trivially holds, which yields

$$TH_0T^* = H_0 \ . \tag{3.22}$$

Further Schrödingers equation yields

$$\begin{aligned} & i\hbar\partial_t|\Psi\rangle = (H_I + H_0)\,|\Psi\rangle \\ \Leftrightarrow \quad & i\hbar T\partial_t T^*|\Psi\rangle = (TH_IT^* + TH_0T^*)\,|\Psi\rangle \\ \Leftrightarrow \quad & i\hbar e^{i/\hbar H_T t}\partial_t e^{-i/\hbar H_T t}|\Psi\rangle = (TH_IT^* + H_0)\,|\Psi\rangle \\ \Leftrightarrow \quad & H_T|\Psi\rangle + i\hbar\partial_t|\Psi\rangle = (TH_IT^* + H_0)\,|\Psi\rangle \\ \Leftrightarrow \quad & i\hbar\partial_t|\Psi\rangle = (TH_IT^* + H_0 - H_T)\,|\Psi\rangle \end{aligned} \tag{3.23}$$

Hence, we define the transformed Hamiltonian of the system H. This operator is given by

$$\begin{aligned} H &:= TH_IT^* + H_0 - H_T \\ &= i\hbar \sum_{J=0}^{N} \sum_{l\in\mathbb{N}} C_l \left(e^{ik_l\cdot x_J} \sigma_{2,1}^{(J)} c_l - e^{-ik_l\cdot x_J} \sigma_{1,2}^{(J)} c_l^\dagger \right) \\ &\quad + \hbar \sum_{J=1}^{N} (\omega - \omega_0)\sigma_{2,1}^{(J)}\sigma_{1,2}^{(J)} + \hbar \sum_{l\in\mathbb{N}} (\omega_l - \omega_0) c_l^\dagger c_l \ . \end{aligned} \tag{3.24}$$

Using Schrödingers equation we find the system of differential equations describing the probability amplitudes. The calculations in section (6.1.3) prove the following:

$$\begin{aligned} \dot{b}(t) &= -\sum_{n\in\mathbb{N}} C_n a_n(t) e^{ik_n\cdot x_0} \\ \dot{b}_j(t) &= -(\omega - \omega_0) b_j(t) - \sum_{n\in\mathbb{N}} C_n a_n(t) e^{ik_n\cdot x_j} && ,\forall j \in \{1, ..., N\} \\ \dot{a}_n(t) &= -(\omega_n - \omega_0) a_n(t) + C_n b(t) e^{-ik_n\cdot x_0} + C_n \sum_{j=1}^{N} b_j(t) e^{-ik_n\cdot x_j} && ,\forall n \in \mathbb{N} \end{aligned} \tag{3.25}$$

3.2 Derivation of Delay Differential Equation Describing the Initially Excited Atom

In this subsection we will derive the delay differential equation describing the initially excited atom at x_0. This section is geared to [2].
The procedure presented in this section is in its idea transferable to Heisenbergs picture. We start by eliminating the differential equation of the electromagnetic field. This is accomplished by calculating the analytical solution of a_n and inserting it into the system of differential equations for b and b_j with $j \in \{1, .., N\}$. Following the restriction of the system to electromagnetic field modes with wave vectors k_n parallel to the z-axis defines the constant C_n and simplifies the equations as the scalar product becomes the multiplication on $\mathbb{R}$. Passing to the limit in which the length L of the quantisation box goes to infinity the differential equations become integro-differential equations. We emphasise that in contrast to most approaches where the coupling to the quantised electromagnetic field is chosen to be constant we keep this frequency depending coupling. This yields a rigorous but more complicated derivation of the refraction coefficient. Further we assume that in the considered system b varies much slower in time than $e^{-i\omega_0 t}$ which will yield a handy representation of the differential equations in the limit where L goes to infinity. Passing to the one excitation limit we can deduce a solvable differential equation for b_j with $j \in \{1, .., N\}$. This solution and the limit of a continuous dielectric medium will yield the final delay differential equation for b with an initial value.

We consider the system of differential equations derived in the previous subsection. We recall

$$\begin{aligned}
\dot{b}(t) &= -\sum_{n\in\mathbb{N}} C_n a_n(t) e^{ik_n\cdot x_0} \\
\dot{b}_j(t) &= -(\omega-\omega_0)b_j(t) - \sum_{n\in\mathbb{N}} C_n a_n(t) e^{ik_n\cdot x_j} && ,\forall j \in \{1, ..., N\} \\
\dot{a}_n(t) &= -(\omega_n-\omega_0)a_n(t) + C_n b(t) e^{-ik_n\cdot x_0} + C_n \sum_{j=1}^{N} b_j(t) e^{-ik_n\cdot x_j} && ,\forall n \in \mathbb{N}
\end{aligned} \tag{3.26}$$

For linear differential equations with constant coefficients *Duhamels formula* gives an explicit representation of the solution. We recall

Theorem 3.2. *(Duhamels formula) Be $A \in \mathcal{C}(I, L(X))$, $b \in \mathcal{C}(I, X)$ and $I \subseteq \mathbb{R}$ an interval, $t_0 \in I$, $u_0 \in X$. Then has the initial value problem*

$$\begin{cases} \dot{u}(t) + A(t)u(t) = b(t) \\ \qquad\qquad u(t_0) = u_0 \ . \end{cases}$$

a unique solution $u \in \mathcal{C}^1(I, X)$ with the representation

$$u(t) = U(t, t_0)u_0 + \int_{t_0}^{t} dt' U(t, t')b(t') \ ,$$

where $U(t,t')$ is the propagator. Be further $A(t) \equiv A$. Then the solution is given by

$$u(t) = e^{-(t-t_0)A} u_0 + \int_{t_0}^{t} dt' e^{(t'-t)A} b(t') \ .$$

[8]

Applying this theorem to the differential equation describing a_n yields

$$a_n(t) = \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} \left(C_n b(t') e^{-ik_n \cdot x_0} + C_n \sum_{j=1}^{N} b_j(t') e^{-ik_n \cdot x_j} \right) \ . \tag{3.27}$$

The justification for the applicability of theorem (3.2) is presented in section (6.1.4). We will now insert this solution into the first two equations of (3.26). This will reduce the number of considered equations. Further the system will be restricted to a system of electromagnetic field modes with wave vectors k_n parallel to the z-axis. This yields the calculations in section (6.1.5) with the final differential equations

$$\dot{b}(t) = -\sum_{n\in\mathbb{N}} C_n^2 \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b(t') - \sum_{n\in\mathbb{N}} \sum_{j=1}^{N} C_n^2 \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b_j(t') e^{ik_n(z_0-z_j)} \tag{3.28}$$

and

$$\begin{aligned} \dot{b}_j(t) = -(\omega-\omega_j) b_j(t) &- \sum_{n\in\mathbb{N}} C_n^2 e^{ik_n(z_j-z_0)} \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b(t') \\ &- \sum_{n\in\mathbb{N}} \sum_{l=1}^{N} C_n^2 e^{ik_n(z_j-z_l)} \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b_l(t') \ . \end{aligned} \tag{3.29}$$

A rigorous but more complicated part of the refraction coefficients derivation is presented in section (6.1.6). As mentioned before, these calculations keep the frequency depending coupling to the quantised electromagnetic field. As the length of the quantised box goes to infinity we obtain the limits

$$\sum_{n\in\mathbb{N}} C_n^2 \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b(t') \longrightarrow \frac{\pi\mu^2\omega_0}{\hbar A c} b(t) \tag{3.30}$$

and

$$\begin{aligned} &\sum_{n\in\mathbb{N}} C_n^2 e^{ik_n(z_0-z_j)} \int_0^t dt' b_j(t') e^{i(\omega_n-\omega_0)(t'-t)} \\ &\longrightarrow \frac{\omega_0\pi\mu^2}{\hbar A c} b_j(t-|z_0-z_j|/c) e^{i\omega_0|z_0-z_j|/c} \Theta(t-|z_0-z_j|/c) \ . \end{aligned} \tag{3.31}$$

Defining the constants

$$K := \frac{\omega_0\pi\mu^2}{\hbar A c}, \qquad l_{jm} = |z_m - z_j|, \ \forall \ m,j \in \mathbb{N}, \qquad \text{and} \qquad l_j = l_{j0} \tag{3.32}$$

we can rewrite the resulting equations as

$$\begin{aligned}
\dot{b}(t) &= -Kb(t) - K\sum_{j=1}^{N} e^{ik_0 l_j} b_j(t - l_j/c)\Theta(t - l_j/c) \\
\dot{b}_j(t) &= -i(\omega - \omega_0)b_j(t) \\
&\quad - Ke^{ik_0 l_j} b(t - l_j/c)\Theta(t - l_j/c) - Kb_j(t) \\
&\quad - K \sum_{m\in\{1,\dots,N\}\setminus\{j\}} e^{ik_0 l_j} b_m(t - l_{jm}/c)e^{ik_0}\Theta(t - l_{jm}/c) \ .
\end{aligned} \tag{3.33}$$

These differential equations describe the atom-atom interaction of the considered system. We see that the interaction with other atoms is delayed in time with $t - l_j/c$ respectively $t - l_{jm}/c$. This delay is caused by the fact that the atoms do not interact directly with each other but through the electromagnetic field.
Passing to the one excitation limit the coupling of the probability amplitudes in the dielectric medium becomes negligible. Using that we consider systems where $|\omega - \omega_0| \gg K$ holds we obtain

$$b_j(t) = \frac{iK}{\omega - \omega_0} e^{ik_0 l_j} b(t - l_j/c)\Theta(t - l_j/c) \ . \tag{3.34}$$

For more detailed calculations see section (6.1.7). Using that this yields

$$b_j(t - l_j/c)\Theta(t - l_j/c) = \frac{iK}{\omega - \omega_0} e^{ik_0 l_j} b(t - 2l_j/c)\Theta(t - 2l_j/c) \tag{3.35}$$

we conclude that

$$\begin{aligned}
\dot{b}(t) &= -Kb(t) - K\sum_{j=1}^{N} e^{ik_0 l_j} b_j(t - l_j/c)\Theta(t - l_j/c) \\
&= -Kb(t) - \frac{iK^2}{\omega - \omega_0}\sum_{j=1}^{N} e^{2ik_0 l_j} b(t - 2l_j/c)\Theta(t - 2l_j/c) \ .
\end{aligned} \tag{3.36}$$

We have reduced the system of differential equations in (3.26) to one differential equation describes the probability amplitude b of the initially excited atom at x_0 under the influence of the dielectric medium.
Up to now the dielectric medium was considered as a discrete set of two-state atoms. We will now pass to the limit in which the dielectric medium is considered as a continuous block with effective area A. This block contains $N\ A\ dz$ atoms in the slice $[z, z + dz]$. This limit yields

$$\begin{aligned}
&NA\int_l^\infty dz' e^{2ik_0(z'-z_0)} b(t - 2(z' - z_0)/c)\Theta(t - 2(z' - z_0)/c) \\
&= -\frac{NA}{2ik_0} e^{2ik_0(l-z_0)} b(t - 2(l - z_0)/c)\Theta(t - 2(l - z_0)/c) \\
&\quad + \frac{NA}{2ik_0} e^{2ik_0(z'-z_0)} b(t - 2(z' - z_0)/c)\Theta(t - 2(z' - z_0)/c)\Big|_{z'=\infty} \ .
\end{aligned} \tag{3.37}$$

The detailed calculations are presented in section (6.1.8). We see that the probability amplitude is delayed by $2(z'-z_0)/c$ and that $2(z'-z_0)/c \to \infty$ as $z' \to \infty$. Hence the delay becomes infinitely big and therefore the influence of an atom in infinitive distance from the atom at x_0 is negligible. Defining $d := l - z_0$ we obtain

$$\begin{aligned} &NA \int_l^\infty dz' e^{2ik_0(z'-z_0)} b(t - 2(z'-z_0)/c)\Theta(t - 2(z'-z_0)/c) \\ &= -\frac{NA}{2ik_0} e^{2ik_0 d} b(t - 2d/c)\Theta(t - 2d/c) \ . \end{aligned} \tag{3.38}$$

Inserting this result into (3.36) leads to

$$\begin{aligned} \dot{b}(t) &= -Kb(t) - \frac{iK^2}{\omega - \omega_0} \sum_{j=1}^{N} e^{2ik_0 l_j} b(t - 2l_j/c)\Theta(t - 2l_j/c) \\ &\longrightarrow -Kb(t) + \frac{i}{\omega - \omega_0} \left(\frac{\pi \mu^2 \omega_0}{\hbar A c} \right)^2 \frac{NA}{2ik_0} e^{2ik_0 d} b(t - 2d/c)\Theta(t - 2d/c) \\ &= -Kb(t) + \frac{1}{\omega - \omega_0} \frac{\pi^2 \mu^4 \omega_0 N}{\hbar^2 A c 2} e^{2ik_0 d} b(t - 2d/c)\Theta(t - 2d/c) \\ &= -Kb(t) + \frac{\pi \mu^2 N}{2\hbar(\omega - \omega_0)} K e^{2ik_0 d} b(t - 2d/c)\Theta(t - 2d/c) \ . \end{aligned} \tag{3.39}$$

We will now analyse the influence that the refraction has on the probability amplitude. We recall that the refraction index of a dielectric medium of N two-state atoms per unit volume is given by

$$n(\omega)^2 - 1 = \frac{4\pi N \mu^2 \omega}{\hbar(\omega^2 - \omega_0^2)} \ . \tag{3.40}$$

Here we assumed that the dielectric medium contains only atoms with the transition frequency ω and transition dipole moment μ. Further we ignored again local field effects in the dielectric medium. We now assume that $n(\omega_0) \approx 1$ and as above that $\omega \approx \omega_0$. Combining these assumptions with (3.40) we find

$$\begin{aligned} \frac{n(\omega_0) - 1}{n(\omega_0) + 1} &= \frac{n(\omega_0)^2 - 1}{(n(\omega_0) + 1)^2} \approx \frac{n(\omega_0)^2 - 1}{4} = \frac{1}{4} \frac{4\pi N \mu^2 \omega}{\hbar(\omega^2 - \omega_0^2)}\bigg|_{\omega=\omega_0} = \frac{\pi N \mu^2 \omega}{\hbar(\omega - \omega_0)(\omega + \omega_0)}\bigg|_{\omega=\omega_0} \\ &\approx \frac{\pi N \mu^2}{2\hbar(\omega - \omega_0)}\bigg|_{\omega=\omega_0} \ . \end{aligned} \tag{3.41}$$

For $n := n(\omega)$ we therefore conclude that

$$R = -\frac{n-1}{n+1} \approx -\frac{\pi N \mu^2}{2\hbar(\omega - \omega_0)} \ , \tag{3.42}$$

where R is the reflection coefficient according to Fresnels formula for normal incidence. Inserting this into (3.39) we find the initial value problem

$$\begin{cases} \dot{b}(t) = -K\Big(b(t) + Re^{2ik_0 d} b(t - 2d/c)\Theta(t - 2d/c)\Big) \\ b(0) = 1 \ . \end{cases} \tag{3.43}$$

This is a delay differential equation with only an initial value. In general an initial value is not enough to describe a unique solution of a delay differential equation. However we will see in the next subsection that in this particular case it is enough.

3.3 Solution of the Delay Differential Equation Describing the Initially Excited Atom

We will now solve the delay differential equation derived in the previous subsection. We remark that this equation is solvable using *Laplace transformation* which is presented in [9]. This approach however will yield a non-trivial root problem with infinite solutions. We will here present an alternative approach, using a decomposition of the $\mathbb{R}_+$ in small intervals.
We consider the delay differential equation derived in the previous subsection. We recall

$$\begin{cases} \dot{b}(t) = -K\left(b(t) + Re^{2ik_0 d} b(t-2d/c)\Theta(t-2d/c)\right) \\ b(0) = 1\ . \end{cases} \tag{3.44}$$

We first define the constants

$$A = K, \qquad B = -KRe^{2ik_0 d}, \qquad \text{and} \qquad \tau = \frac{2d}{c}\ . \tag{3.45}$$

Therewith we can rewrite (3.44) as

$$\begin{cases} \dot{b}(t) + Ab(t) = Bb(t-\tau)\Theta(t-\tau) \\ b(0) = 1\ . \end{cases} \tag{3.46}$$

We now consider $t \in [0, \tau)$. Then $\Theta(t-\tau) = 0$ and we obtain the differential equation

$$\begin{cases} \dot{b}(t) + Ab(t) = 0 \\ b(0) = 1 \end{cases} \tag{3.47}$$

with the solution

$$b(t) = e^{-At}\ . \tag{3.48}$$

Now we consider $t \in [\tau, 2\tau)$. Then $\Theta(t-\tau) = 1$ and $t-\tau \in [0,\tau)$. Hence we know that $b(t-\tau) = e^{-A(t-\tau)}$ for $t \in [\tau, 2\tau)$. This yields the differential equation

$$\begin{cases} \dot{b}(t) + Ab(t) = Be^{-A(t-\tau)} \\ b(\tau) = e^{-A\tau}\ . \end{cases} \tag{3.49}$$

Duhamels formula leads to

$$b(t) = e^{-(t-\tau)A}e^{-A\tau} + \int_\tau^t dt' e^{(t'-t)A} B e^{-A(t'-\tau)} = e^{-tA} + Be^{-A(t-\tau)}(t-\tau)\ . \tag{3.50}$$

We see that the solution is given by the solution on the interval $[0, \tau)$ plus some extra term. This can be expressed using the Heaviside function as

$$b(t) = e^{-tA} + Be^{A(\tau - t)}(t - \tau)\Theta(t - \tau), \quad t \in [0, 2\tau) \ . \tag{3.51}$$

We assume that the solution of (3.44) on $[n\tau, (n+1)\tau)$ is given by

$$b(t) = \sum_{k=0}^{n} \frac{B^k}{k!}(t - k\tau)^k e^{-A(t-k\tau)}\Theta(t - k\tau) \ . \tag{3.52}$$

We will proof this statement by induction

Induction basis: We have already proven that this formula holds for $n = 0$ (3.48) and $n = 1$ (3.50).

Induction step: We now assume that the statement is true for $(n-1) \in \mathbb{N}$ arbitrary but fixed and $n \geq 2$. We consider the interval $[n\tau, (n+1)\tau)$. We see that $t-\tau \in [(n-1)\tau, n\tau)$. Hence $\Theta(t - \tau) = 1$ and

$$b(t - \tau) = \sum_{k=0}^{n-1} \frac{B^k}{k!}(t - (k+1)\tau)^k e^{-A(t-(k+1)\tau)}\Theta(t - (k+1)\tau) \ . \tag{3.53}$$

The differential equation on the interval $[n\tau, (n+1)\tau)$ is therefore given by

$$\begin{cases} \dot{b}(t) + Ab(t) = \displaystyle\sum_{k=0}^{n-1} \frac{B^{k+1}}{k!}(t - (k+1)\tau)^k e^{-A(t-(k+1)\tau)}\Theta(t - k\tau) \\ b(n\tau) = \displaystyle\sum_{k=0}^{n-1} \frac{B^k}{k!}(n\tau - k\tau)^k e^{-A(n\tau-k\tau)}\Theta(n\tau - k\tau) \ . \end{cases} \tag{3.54}$$

Duhamels formula leads to

$$\begin{aligned}
b(t) &= e^{-(t-\tau n)A} b(n\tau) + \int_{n\tau}^{t} dt' e^{(t'-t)A} \sum_{k=0}^{n-1} \frac{B^{k+1}}{k!} (t'-(k+1)\tau)^k e^{-A(t'-(k+1)\tau)} \\
&= e^{-(t-\tau n)A} b(n\tau) + \sum_{k=0}^{n-1} \frac{B^{k+1}}{k!} e^{-A(t-(k+1)\tau)} \int_{n\tau}^{t} dt' (t'-(k+1)\tau)^k \\
&= \sum_{k=0}^{n-1} \frac{B^{k+1}}{(k+1)!} (t-(k+1)\tau)^{k+1} e^{-A(t-(k+1)\tau)} \\
&\quad + e^{-(t-\tau n)A} b(n\tau) - \sum_{k=0}^{n-1} \frac{B^{k+1}}{(k+1)!} e^{-A(t-(k+1)\tau)} (n\tau-(k+1)\tau)^{k+1} \\
&= \sum_{k=1}^{n} \frac{B^k}{k!} e^{-A(t-k\tau)} (t-\tau)^k \\
&\quad + e^{-(t-\tau n)A} \sum_{k=0}^{n-1} \frac{B^k}{k!} e^{-A(n\tau-k\tau)} (n\tau-k\tau)^k - \sum_{k=1}^{n} \frac{B^k}{k!} e^{-A(t-k\tau)} (n\tau-k\tau)^k \\
&= \sum_{k=1}^{n} \frac{B^k}{k!} (t-\tau)^k e^{-A(t-k\tau)} + e^{-At} \\
&= \sum_{k=0}^{n} \frac{B^k}{k!} (t-\tau)^k e^{-A(t-k\tau)} = \sum_{k=0}^{n} \frac{B^k}{k!} (t-\tau)^k e^{-A(t-k\tau)} \Theta(t-k\tau) \ ,
\end{aligned} \tag{3.55}$$

which is the statement.
As $\{ \ [n\tau, (n+1)\tau) \ | n \in \mathbb{N} \ \}$ covers $\mathbb{R}_+$. The solution of (3.46) for $t \in \mathbb{R}_+$ is given by

$$b(t) = \sum_{k=0}^{\infty} \frac{B^k}{k!} (t-k\tau)^k e^{-A(t-k\tau)} \Theta(t-k\tau) \ . \tag{3.56}$$

Therefore the solution of (3.44) is given by

$$b(t) = \sum_{n=0}^{\infty} \frac{\left(-KRe^{2ik_0 d}\right)^n}{n!} \left(t - \frac{2nd}{c}\right)^n e^{-K(t-2nd/c)} \Theta\left(t - \frac{2nd}{c}\right) \ . \tag{3.57}$$

4 Description of Atom-Field Interaction Using Heisenbergs Picture

One aim of this thesis is to lay a foundation for the description of a system of a two state atom with an electromagnetic field and a dielectric medium consisting out of partially excited two state atoms. This energetic gain of the system will lead to a system in which the number of excitations is bigger than one and especially not fixed. Hence the transition into Heisenbergs picture is inevitable. Describing the dynamics of the system using the probability amplitudes will lead to not tractable problems in the derivation of a solution for the system of differential equations.
We will therefore transfer into Heisenbergs picture where the dynamics of the system is described using the operators of the Hamiltonian. This will yield differential equations of functions from $\mathbb{R}_+$ to $L(\mathscr{H})$ the space of linear and bounded operators on $\mathscr{H}$. We will see that the physical interpretation of these operator equations is much more difficult than the interpretation of probability amplitudes. Further we will describe the mathematical framework how integration on $L(\mathscr{H})$ is to be understood. We will demonstrate the general problematic in Heisenbergs picture which is the non negligible *noise term*.

4.1 Derivation of Differential Operator Equations Describing the Atom-Field Interaction

We start with the in section 3.1 derived transformed atom-field interaction Hamiltonian

$$\begin{aligned} H = \underbrace{i\hbar \sum_{j=0}^{N} \sum_{n\in\mathbb{N}} C_n \left(e^{ik_n \cdot x_j} \sigma_{2,1}^{(j)} c_n - e^{-ik_n \cdot x_j} \sigma_{1,2}^{(j)} c_n^\dagger \right)}_{=:H_I} \\ + \hbar \sum_{j=1}^{N} (\omega - \omega_0) \sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)} + \hbar \sum_{n\in\mathbb{N}} (\omega_n - \omega_0) c_n^\dagger c_n \ . \end{aligned} \tag{4.1}$$

To derive differential operator equations for the given system we will use Heisenbergs equation which states

$$\frac{dA}{dt} = \frac{i}{\hbar} [H, A] + \partial_t A \ . \tag{4.2}$$

In our case the operators $\sigma_{2,1}^{(j)}$ and $c_n^\dagger$ are not implicitly time dependant and therefore $\partial_t \sigma_{2,1}^{(j)} = \partial_t c_n^\dagger = 0$. We will now analyse the commutator relation of the Hamiltonian and

$\sigma_{2,1}^{(J)}$ for $J \in \{0, ..., N\}$ arbitrary but fixed. We find

$$\begin{aligned}
[H_I, \sigma_{2,1}^{(J)}] &= i\hbar \sum_{j=0}^{N} \sum_{n\in\mathbb{N}} C_n e^{ik_n\cdot x_j} [\sigma_{2,1}^{(j)} c_n, \sigma_{2,1}^{(J)}] - i\hbar \sum_{j=0}^{N} \sum_{n\in\mathbb{N}} C_n e^{-ik_n\cdot x_j} [\sigma_{1,2}^{(j)} c_n^\dagger, \sigma_{2,1}^{(J)}] \\
&= -i\hbar \sum_{j=0}^{N} \sum_{n\in\mathbb{N}} C_n e^{-ik_n\cdot x_j} [\sigma_{1,2}^{(j)}, \sigma_{2,1}^{(J)}] c_n^\dagger = -i\hbar \sum_{n\in\mathbb{N}} C_n e^{-ik_n\cdot x_J} \left(\sigma_{1,1}^{(J)} - \sigma_{2,2}^{(J)}\right) c_n^\dagger \ .
\end{aligned} \tag{4.3}$$

For the other terms we obtain

$$\begin{aligned}
&\hbar \sum_{j=1}^{N} (\omega - \omega_0) [\sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)}, \sigma_{2,1}^{(J)}] + \hbar \sum_{n\in\mathbb{N}} (\omega_n - \omega_0) [c_n^\dagger c_n, \sigma_{2,1}^{(J)}] \\
&\quad = \hbar \sum_{j=1}^{N} (\omega - \omega_0) \delta_{J,j} \sigma_{2,1}^{(J)} [\sigma_{1,2}^{(j)}, \sigma_{2,1}^{(j)}] = \hbar(\omega - \omega_0) \sigma_{2,1}^{(J)} \left(\sigma_{1,1}^{(J)} - \sigma_{2,2}^{(J)}\right) \\
&\quad = \hbar(\omega - \omega_0) \sigma_{2,1}^{(J)}
\end{aligned} \tag{4.4}$$

where $J \in \{1, ..., N\}$. We proceed equivalently for the operator $c_L^\dagger$ which leads to

$$\begin{aligned}
[H_I, c_L^\dagger] &= i\hbar \sum_{j=0}^{N} \sum_{n\in\mathbb{N}} C_n e^{ik_n\cdot x_j} [\sigma_{2,1}^{(j)} c_n, c_L^\dagger] - i\hbar \sum_{j=0}^{N} \sum_{n\in\mathbb{N}} C_n e^{-ik_n\cdot x_j} [\sigma_{1,2}^{(j)} c_n^\dagger, c_L^\dagger] \\
&= i\hbar \sum_{j=0}^{N} \sum_{n\in\mathbb{N}} C_n e^{ik_n\cdot x_j} \sigma_{2,1}^{(j)} [c_n, c_L^\dagger] = i\hbar \sum_{j=0}^{N} C_n e^{ik_L\cdot x_j} \sigma_{2,1}^{(j)} \ .
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
&\hbar \sum_{j=1}^{N} (\omega - \omega_0) [\sigma_{2,1}^{(j)} \sigma_{1,2}^{(j)}, c_L^\dagger] + \hbar \sum_{n\in\mathbb{N}} (\omega_n - \omega_0) [c_n^\dagger c_n, c_L^\dagger] \\
&\quad = \hbar \sum_{n\in\mathbb{N}} (\omega_n - \omega_0) \delta_{L,n} c_L^\dagger [c_n, c_n^\dagger] = \hbar(\omega_L - \omega_0) c_L^\dagger \ .
\end{aligned} \tag{4.6}$$

Inserting these results into Heisenbergs equation (4.2) we find the following system of differential equations

$$\begin{cases}
\dfrac{d}{dt}\sigma_{2,1} = \displaystyle\sum_{n\in\mathbb{N}} C_n e^{-ik_n\cdot x_0} c_n^\dagger \left(\sigma_{1,1} - \sigma_{2,2}\right) \\
\dfrac{d}{dt}\sigma_{2,1}^{(j)} = i(\omega - \omega_0)\sigma_{2,1}^{(j)} + \displaystyle\sum_{n\in\mathbb{N}} C_n e^{-ik_n\cdot x_j} c_n^\dagger \left(\sigma_{1,1}^{(j)} - \sigma_{2,2}^{(j)}\right) & , j \in \{1, ..., N\} \\
\dfrac{d}{dt}c_n^\dagger = i(\omega_n - \omega_0) c_n^\dagger - C_n e^{ik_n\cdot x_0} \sigma_{2,1} - \displaystyle\sum_{j=1}^{N} C_n e^{ik_n\cdot x_j} \sigma_{2,1}^{(j)} & , n \in \mathbb{N}
\end{cases} \tag{4.7}$$

To emphasise the analogy to Schrödingers picture we have separated the differential equation describing the initially excited atom at x_0 from those describing the dielectric medium. We have further omitted the index $j = 0$ for this particular atom.

4.2 Derivation of Delay Differential Operator Equation Describing the Initially Excited Atom

We will now use Duhamels formula for the differential operator equation describing $c_n^\dagger$. We stress that the here used integral is different to the Lebesgue integral used in previous sections. We here use its generalisation to Banach space valued function the so called *Bochner-Integral*. We interpret a time dependant operator as a map that relates the time to an operator. We recall that any Fockspace $\mathscr{H}$ is by definition a Hilbert space. Hence $(\mathscr{H}, ||\cdot||_{\mathscr{H}})$ with $||\cdot||_{\mathscr{H}}$ the scalar product induced norm on $\mathscr{H}$ is a Banach space. Therefore the operator space $(L(\mathscr{H}), ||\cdot||)$ with the operator norm is a Banach space itself. Hence a map that relates the time to an operator in $(L(\mathscr{H}), ||\cdot||)$ is a Banach space valued function. Therefore the Bochner-Integral is well-defined and we obtain

$$\begin{aligned} c_n^\dagger(t) = e^{i(\omega_n-\omega_0)t} c_n^\dagger(0) - \int_0^t dt'\ e^{-i(\omega_n-\omega_0)(t'-t)} C_n e^{ik_n\cdot x_0} \sigma_{2,1}(t') \\ - \sum_{j=1}^{N} \int_0^t dt'\ e^{-i(\omega_n-\omega_0)(t'-t)} C_n e^{ik_n\cdot x_j} \sigma_{2,1}^{(j)}(t') \ . \end{aligned} \tag{4.8}$$

Using this explicit representation of $c_n^\dagger$ we rewrite the system differential equations (4.7). For the operator $\sigma_{2,1}$ we find

$$\begin{aligned} \frac{d}{dt}\sigma_{2,1}(t) = &- \sum_{n\in\mathbb{N}} C_n^2 \int_0^t dt'\ e^{-i(\omega_n-\omega_0)(t'-t)} \sigma_{2,1}(t') \left(\sigma_{1,1}(t) - \sigma_{2,2}(t)\right) \\ &- \sum_{n\in\mathbb{N}} \sum_{j=1}^{N} C_n^2 e^{-ik_n\cdot(x_0-x_j)} \int_0^t dt'\ e^{-i(\omega_n-\omega_0)(t'-t)} \sigma_{2,1}^{(j)}(t') \left(\sigma_{1,1}(t) - \sigma_{2,2}(t)\right) \\ &+ \sum_{n\in\mathbb{N}} C_n e^{-ik_n\cdot x_0} e^{i(\omega_n-\omega_0)t} c_n^\dagger(0) \left(\sigma_{1,1} - \sigma_{2,2}\right) \ . \end{aligned} \tag{4.9}$$

We emphasise that in contrast to typical approaches we will keep the frequency dependant coupling to the electromagnetic field in the following. We restrict our system to modes of the electromagnetic field with wave vectors k_n parallel to the z-axis. Thus we take

$$C_n = \frac{\mu}{\hbar} \left(\frac{2\pi\hbar\omega_n}{AL}\right)^{\frac{1}{2}} \ , \tag{4.10}$$

where A is an effective area, L is the length along the z-axis of the quantised box and μ is the magnitude of the transition dipole moment of each two-state atom. Inserting this into the differential operator equation yields

$$\begin{aligned}\frac{d}{dt}\sigma_{2,1}(t) = &-\int_0^t dt'\ \frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{-i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}(t')\left(\sigma_{1,1}(t)-\sigma_{2,2}(t)\right)\\ &-\sum_{j=1}^{N}\int_0^t dt'\ \frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{-ik_n\cdot(z_0-z_j)}e^{-i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}^{(j)}(t')\left(\sigma_{1,1}(t)-\sigma_{2,2}(t)\right)\\ &+\sum_{n\in\mathbb{N}}\frac{\mu}{\hbar}\left(\frac{2\pi\hbar\omega_n}{AL}\right)^{\frac{1}{2}}e^{-ik_n\cdot z_0}e^{i(\omega_n-\omega_0)t}c_n^\dagger(0)\left(\sigma_{1,1}(t)-\sigma_{2,2}(t)\right)\ .\end{aligned} \tag{4.11}$$

We recall that we consider a closed system in which the atom at x_0 is initially excited. Hence, for $t \neq 0$ there can be only one atom excited in the dielectric medium. Assuming N large enough the assumptions

$$\left\langle\sigma_{1,1}^{(j)}(t)\right\rangle = 1 \quad ,\forall j\in\{1,...,N\} \tag{4.12}$$

and

$$\left\langle\sigma_{2,2}^{(j)}(t)\right\rangle = 0 \quad ,\forall j\in\{1,...,N\} \tag{4.13}$$

are justified as at most one atom in the dielectric medium can be excited and therefore the occupation in average is close to zero. We mention that for a controlled system via gain, this assumption does not hold true.

Analogously to previous calculations we can insert (4.8) into the second equation of (4.7) which yields

$$\begin{aligned}\frac{d}{dt}\sigma_{2,1}^{(j)} = &\, i(\omega-\omega_0)\sigma_{2,1}^{(j)} - \int_0^t dt'\ \frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}^{(j)}(t')\\ &-\int_0^t dt'\ \frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{-ik_n\cdot(z_j-z_0)}e^{-i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}(t')\\ &-\sum_{J\in\{1,...,N\}\setminus\{j\}}\int_0^t dt'\ \frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{-ik_n\cdot(z_j-z_J)}e^{-i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}^{(J)}(t')\\ &+\sum_{n\in\mathbb{N}}\frac{\mu}{\hbar}\left(\frac{2\pi\hbar\omega_n}{AL}\right)^{\frac{1}{2}}e^{-ik_n\cdot z_j}e^{-i(\omega_n-\omega_0)t}c_n^\dagger(0)\ .\end{aligned} \tag{4.14}$$

We will now rewrite the differential equations (4.11) and (4.14) using the limit in which the length L of the quantisation box goes to infinity. The detailed calculations are given in section (6.2.1). They yield

$$\int_0^t dt'\ \frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{-i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}(t') \longrightarrow \frac{\pi\mu^2\omega_0}{\hbar Ac}\sigma_{2,1}(t) \tag{4.15}$$

and

$$
\begin{aligned}
&\int_0^t dt' \ \frac{2\pi\mu^2}{AL\hbar} \sum_{n\in\mathbb{N}} \omega_n e^{-ik_n Z} e^{-i(\omega_n-\omega_0)(t'-t)} \sigma_{2,1}^{(J)}(t') \\
&\longrightarrow \frac{i\pi\mu^2}{\hbar Ac} \left(\int_0^t dt' \ e^{i\omega_0(t'-t)} \sigma_{2,1}^{(J)}(t') \frac{d}{dt'} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' e^{-i\omega'(t'-t+Z/c)} \right. \\
&\qquad\qquad \left. + \int_0^t dt' \ e^{i\omega_0(t'-t)} \sigma_{2,1}^{(J)}(t') \frac{d}{dt'} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' e^{-i\omega'(t'-t-Z/c)} \right) \\
&= \frac{\omega_0\pi\mu^2}{\hbar Ac} e^{-i\omega_0|Z|/c_0} \sigma_{2,1}^{(J)}(t-|Z|/c_0)\Theta(t-|z_0-z_J|/c_0) \ .
\end{aligned}
\tag{4.16}
$$

In contrast to Schrödingers picture we have to consider the noise term in (4.14). In the same limit as above we obtain

$$
\begin{aligned}
&\sum_{n\in\mathbb{N}} \frac{\mu}{\hbar} \left(\frac{2\pi\hbar\omega_n}{AL} \right)^{\frac{1}{2}} e^{-i(k_n z_0 + (\omega_0-\omega_n)t)} c_n^\dagger(0) \\
&\qquad \longrightarrow \frac{\mu}{\hbar} \left(\frac{2\pi\hbar}{AL} \right)^{\frac{1}{2}} \frac{L}{2\pi} \int_{-\infty}^{\infty} dk' \ (\omega')^{\frac{1}{2}} e^{-i(k'z_0+(\omega_0-\omega')t)} c_{k'}^\dagger(0) \ .
\end{aligned}
\tag{4.17}
$$

Defining $l_J := |z_0 - z_J|$, $K := \pi\mu^2\omega_0/(\hbar Ac)$ and

$$
\Delta B(j,t) = \left(\frac{AL\hbar}{2\pi} \right)^{\frac{1}{2}} \frac{1}{\mu\pi k_0} \int_{-\infty}^{\infty} dk' \ (\omega')^{\frac{1}{2}} e^{-i(k'z_j+(\omega_0-\omega')t)} c_{k'}^\dagger(0) \tag{4.18}
$$

we obtain

$$
\begin{aligned}
\frac{d}{dt}\sigma_{2,1}(t) = &-K\sigma_{2,1}(t) \\
&- K \sum_{j=1}^{N} e^{-ik_0 l_j} \sigma_{2,1}^{(j)}(t-l_j/c_0)\Theta(t-l_j/c_0) \\
&+ K\Delta B(0,t) \ .
\end{aligned}
\tag{4.19}
$$

In the same way we find with $l_{j,J} := |z_j - z_J|$ that

$$
\begin{aligned}
\frac{d}{dt}\sigma_{2,1}^{(j)}(t) = \ & i(\omega-\omega_0)\sigma_{2,1}^{(j)} - K\sigma_{2,1}^{(j)}(t) - Ke^{-ik_0 l_j}\sigma_{2,1}(t-l_j/c_0)\Theta(t-l_j/c_0) \\
&- K \sum_{J\in\{1,\ldots,N\}\setminus\{j\}} e^{-ik_0 l_{j,J}} \sigma_{2,1}^{(J)}(t-l_{j,J}/c_0)\Theta(t-l_{j,J}/c_0) \\
&+ K\Delta B(j,t) \ .
\end{aligned}
\tag{4.20}
$$

As we suppose that the atom in x_0 is initially excited, we neglect the interaction between the atoms in the dielectric medium which is described by the term

$$
W_{int} := K \sum_{J\in\{1,\ldots,N\}\setminus\{j\}} e^{-ik_0 l_{j,J}} \sigma_{2,1}^{(J)}(t-l_{j,J}/c_0)\Theta(t-l_{j,J}/c_0) \ . \tag{4.21}
$$

This yields the differential equation

$$\begin{aligned}\frac{d}{dt}\sigma_{2,1}^{(j)}(t) &= -\left(i(\omega_0-\omega)+K\right)\sigma_{2,1}^{(j)}(t) - Ke^{-ik_0l_j}\sigma_{2,1}(t-l_j/c_0)\Theta(t-l_j/c_0) + K\Delta B(j,t)\\ &\approx -i(\omega_0-\omega)\sigma_{2,1}^{(j)}(t) - Ke^{-ik_0l_j}\sigma_{2,1}(t-l_j/c_0)\Theta(t-l_j/c_0) + K\Delta B(j,t)\ ,\end{aligned} \tag{4.22}$$

as $|\omega-\omega_0| \gg K$.
For this differential equation Duhamels formula is applicable and yields

$$\begin{aligned}\sigma_{2,1}^{(j)}(t) = e^{-i(\omega_0-\omega)t}\sigma_{2,1}^{(j)}(0) &- Ke^{-ik_0l_j}\int_0^t dt' e^{i(\omega_0-\omega)(t'-t)}\sigma_{2,1}(t'-l_j/c_0)\Theta(t'-l_j/c_0)\\ &+ \int_0^t dt' e^{i(\omega_0-\omega)(t'-t)}K\Delta B(j,t')\ .\end{aligned} \tag{4.23}$$

By analysing the integral describing the delay contribution of $\sigma_{2,1}$, which can be seen in section (6.2.2), we find

$$\begin{aligned}\sigma_{2,1}^{(j)}(t) = e^{-i(\omega_0-\omega)t}&\sigma_{2,1}^{(j)}(0)\\ &+ i\frac{K}{(\omega_0-\omega)}e^{-ik_0l_j}\left(\sigma_{2,1}(t-l_j/c_0) - e^{-i(\omega_0-\omega)(t-l_j/c_0)}\sigma_{2,1}(0)\right)\Theta(t-l_j/c_0)\\ &+ \int_0^t dt' e^{-i(\omega_0-\omega)(t'-t)}K\Delta B(j,t')\ .\end{aligned} \tag{4.24}$$

This yields

$$\begin{aligned}\sigma_{2,1}^{(j)}(t-l_j/c_0) = e^{-i(\omega_0-\omega)(t-l_j/c_0)}&\sigma_{2,1}^{(j)}(0)\\ &+ i\frac{K}{(\omega_0-\omega)}e^{-ik_0l_j}\left(\sigma_{2,1}(t-2l_j/c_0)\right.\\ &\quad\left. - e^{-i(\omega_0-\omega)(t-2l_j/c_0)}\sigma_{2,1}(0)\right)\Theta(t-2l_j/c_0)\\ &+ \int_0^{t-l_j/c_0} dt' e^{-i(\omega_0-\omega)(t'-t)}K\Delta B(j,t')\ .\end{aligned} \tag{4.25}$$

As further

$$\Theta(t-l_j/c_0)\Theta(t-2l_j/c_0) = \Theta(t-2l_j/c_0) \tag{4.26}$$

holds we conclude with (4.19) a differential operator equation describing $\sigma_{2,1}$

$$\begin{aligned}\frac{d}{dt}\sigma_{2,1}(t) &= -K\sigma_{2,1}(t) - i\frac{K^2}{(\omega_0-\omega)}\sum_{j=1}^{N} e^{-2ik_0 l_j}\sigma_{2,1}(t-2l_j/c_0)\Theta(t-2l_j/c_0)\left(\sigma_{1,1}(t)-\sigma_{2,2}(t)\right)\\ &- K\sum_{j=1}^{N} e^{-ik_0 l_j} e^{-i(\omega_0-\omega)(t-l_j/c_0)}\sigma_{2,1}^{(j)}(0)\Theta(t-l_j/c_0)\left(\sigma_{1,1}(t)-\sigma_{2,2}(t)\right)\\ &+ i\frac{K^2}{(\omega_0-\omega)}\sum_{j=1}^{N} e^{-2ik_0 l_j} e^{-i(\omega_0-\omega)(t-2l_j/c_0)}\sigma_{2,1}(0)\Theta(t-2l_j/c_0)\left(\sigma_{1,1}(t)-\sigma_{2,2}(t)\right)\\ &- K\sum_{j=1}^{N} e^{-ik_0 l_j}\int_0^{t-l_j/c_0} dt' e^{-i(\omega_0-\omega)(t'-t)} K\Delta B(j,t')\Theta(t-l_j/c_0)\left(\sigma_{1,1}(t)-\sigma_{2,2}(t)\right)\\ &+ K\Delta B(0,t)\left(\sigma_{1,1}(t)-\sigma_{2,2}(t)\right)\ .\end{aligned} \tag{4.27}$$

We pass to the limit of a continuous dielectric medium. The detailed calculation given in section (6.2.3) verifies that

$$\begin{aligned}&\sum_{j=1}^{N} e^{-i2k_0 l_j}\sigma_{2,1}(t-2l_j/c_0)\Theta(t-2l_j/c_0)\\ &\quad\longrightarrow -i\frac{NA}{2k_0}\sigma_{2,1}(t-2(l-z_0)/c_0)\Theta(t-2(l-z_0)/c_0)e^{-i2k_0(l-z_0)}\ .\end{aligned} \tag{4.28}$$

Passing on to the one excitation limit yields that all operator products are in normal form. Hence the noise terms are negligible. This yields

$$\frac{d}{dt}\sigma_{2,1}(t) = -K\sigma_{2,1}(t) - \frac{K^2}{(\omega_0-\omega)}\frac{NA}{2k_0}\sigma_{2,1}(t-2d/c_0)\Theta(t-2d/c_0)e^{-i2k_0 d}\left(\sigma_{1,1}(t)-\sigma_{2,2}(t)\right) \tag{4.29}$$

with $d := l - z_0$. Analogously to (3.40) and following we find that

$$\frac{NAK^2}{2k_0(\omega_0-\omega)} = \frac{\pi\mu^2 N}{2\hbar(\omega_0-\omega)}K \tag{4.30}$$

and

$$\frac{n(\omega_0)-1}{n(\omega_0)+1} \approx \left.\frac{\pi N\mu^2}{2\hbar(\omega-\omega_0)}\right|_{\omega=\omega_0}\ . \tag{4.31}$$

For $n := n(\omega)$ we therefore conclude that

$$R = -\frac{n-1}{n+1} \approx -\frac{\pi N\mu^2}{2\hbar(\omega-\omega_0)}\ , \tag{4.32}$$

where R is the reflection coefficient according to the Fresnel formula for normal incidence. Inserting this into (4.29) we find the delay differential operator equation

$$\frac{d}{dt}\sigma_{2,1}(t) = -K\left(\sigma_{2,1}(t) + Re^{-i2k_0 d}\sigma_{2,1}(t-2d/c_0)\Theta(t-2d/c_0)\left(\sigma_{1,1}(t)-\sigma_{2,2}(t)\right)\right)\ . \tag{4.33}$$

4.3 Analytical Solution of the Operator $\sigma_{2,2}$ and Its Numerical Simulation

We start with the delay differential operator equation derived in the previous subsection. We recall

$$\frac{d}{dt}\sigma_{2,1}(t) = -K\left(\sigma_{2,1}(t) + Re^{-i2k_0d}\sigma_{2,1}(t-\tau)\Theta(t-\tau)\left(\sigma_{1,1}(t) - \sigma_{2,2}(t)\right)\right) \ . \quad (4.34)$$

It is dependant of the operator $\sigma_{2,2}$. We therefore calculate the solution of this operator. The idea is similar to the calculations given in the subsection (3.3). Hence a solution on $\mathbb{R}_+$ can be derived by induction. By definition of the flip operator

$$\sigma_{2,2}(t) = \sigma_{2,1}(t)\sigma_{1,2}(t) \qquad \text{and} \qquad \sigma_{1,1}(t) = 1 - \sigma_{2,2}(t) \quad (4.35)$$

holds true. Hence

$$\begin{aligned}\frac{d}{dt}\sigma_{2,2}(t) &= \left(\frac{d}{dt}\sigma_{2,1}(t)\right)\sigma_{1,2}(t) + \sigma_{2,1}(t)\left(\frac{d}{dt}\sigma_{1,2}(t)\right)\\ &= \left(-K\left(\sigma_{2,1}(t) + Re^{-i2k_0d}\sigma_{2,1}(t-\tau)\Theta(t-\tau)\left(1-2\sigma_{2,2}(t)\right)\right)\right)\sigma_{1,2}(t)\\ &+ \sigma_{2,1}(t)\left(-K\left(\sigma_{1,2}(t) + Re^{i2k_0d}\left(1-2\sigma_{2,2}(t)\right)\sigma_{1,2}(t-\tau)\Theta(t-\tau)\right)\right)\\ &= -K\left(\sigma_{2,2}(t) + Re^{-i2k_0d}\sigma_{2,1}(t-\tau)\Theta(t-\tau)\sigma_{1,2}(t)\right)\\ &- K\left(\sigma_{2,2}(t) + Re^{i2k_0d}\sigma_{2,1}(t)\sigma_{1,2}(t-\tau)\Theta(t-\tau)\right) \ .\end{aligned} \quad (4.36)$$

Due to simplicity of the notation we define

$$\Gamma = -KRe^{-i2k_0d} \quad (4.37)$$

and obtain

$$\frac{d}{dt}\sigma_{2,2}(t) = -2K\sigma_{2,2}(t) + \left(\Gamma\sigma_{2,1}(t-\tau)\sigma_{1,2}(t) + \Gamma^*\sigma_{2,1}(t)\sigma_{1,2}(t-\tau)\right)\Theta(t-\tau) \ . \quad (4.38)$$

As the initial value is only given for the expected value of the operator $\langle\sigma_{2,2}(0)\rangle = 1$ we pass to the equation describing the expected value. As

$$\langle\Gamma\sigma_{2,1}(t-\tau)\sigma_{1,2}(t)\Theta(t-\tau)\rangle = \langle\Gamma^*\sigma_{2,1}(t)\sigma_{1,2}(t-\tau)\Theta(t-\tau)\rangle^* \quad (4.39)$$

holds true we obtain

$$\frac{d}{dt}\langle\sigma_{2,2}(t)\rangle = -2K\langle\sigma_{2,2}(t)\rangle + 2Re\left(\Gamma\langle\sigma_{2,1}(t-\tau)\sigma_{1,2}(t)\Theta(t-\tau)\rangle\right) \ . \quad (4.40)$$

The expected value

$$\langle\sigma_{2,1}(t-\tau)\sigma_{1,2}(t)\Theta(t-\tau)\rangle \quad (4.41)$$

is unknown and therefore problematic. We start by considering $t \in [0, \tau)$. This yields the differential equations

$$\begin{cases} \frac{d}{dt}\langle\sigma_{2,2}(t)\rangle = -2K\langle\sigma_{2,2}(t)\rangle \\ \langle\sigma_{2,2}(0)\rangle = 1 \end{cases} \tag{4.42}$$

and

$$\begin{cases} \frac{d}{dt}\langle\sigma_{2,1}(t)\rangle = -K\langle\sigma_{2,1}(t)\rangle \\ \langle\sigma_{2,1}(0)\rangle = 0 \end{cases} \tag{4.43}$$

This yields immediately

$$\langle\sigma_{2,2}(t)\rangle = e^{-2Kt}, \qquad \text{for } t \in [0, \tau) \tag{4.44}$$

and

$$\langle\sigma_{2,1}(t)\rangle = 0, \qquad \text{for } t \in [0, \tau) \ . \tag{4.45}$$

We further analyse the term

$$\frac{d}{dt}\langle\sigma_{2,1}(0)\sigma_{1,2}(t)\rangle = \langle\sigma_{2,1}(0)\left(\frac{d}{dt}\sigma_{1,2}(t)\right)\rangle = -K\langle\sigma_{2,1}(0)\sigma_{1,2}(t)\rangle \tag{4.46}$$

for $t \in [0, \tau)$ and obtain

$$\langle\sigma_{2,1}(0)\sigma_{1,2}(t)\rangle = e^{-Kt}\langle\sigma_{2,1}(0)\sigma_{1,2}(0)\rangle = e^{-Kt}\langle\sigma_{2,2}(0)\rangle = e^{-Kt} \ . \tag{4.47}$$

We now assume that $t \in [\tau, 2\tau)$. Then we know that $\Theta(t-\tau) = 1$ hence

$$\begin{aligned} &\frac{d}{dt}\langle\sigma_{2,1}(t-\tau)\sigma_{1,2}(t)\rangle \\ &= \langle\sigma_{2,1}(t-\tau)\frac{d}{dt}\sigma_{1,2}(t)\rangle + \langle\left(\frac{d}{dt}\sigma_{2,1}(t-\tau)\right)\sigma_{1,2}(t)\rangle \\ &= -K\langle\sigma_{2,1}(t-\tau)\left(\sigma_{1,2}(t) + Re^{i2k_0d}\left(1-2\sigma_{2,2}(t)\right)\sigma_{1,2}(t-\tau)\Theta(t-\tau)\right)\rangle \\ &\quad - K\langle\sigma_{2,1}(t-\tau)\sigma_{1,2}(t)\rangle \\ &= -2K\langle\sigma_{2,1}(t-\tau)\sigma_{1,2}(t)\rangle + \Gamma^*\langle\sigma_{2,2}(t-\tau)\rangle \\ &\quad - 2\Gamma^*\langle\sigma_{2,1}(t-\tau)\sigma_{2,2}(t)\sigma_{1,2}(t-\tau)\rangle \\ &= -2K\langle\sigma_{2,1}(t-\tau)\sigma_{1,2}(t)\rangle + \Gamma^* e^{-2K(t-\tau)} \ . \end{aligned} \tag{4.48}$$

Duhamels formula leads to

$$\begin{aligned} \langle\sigma_{2,1}(t-\tau)\sigma_{1,2}(t)\rangle &= e^{-(t-\tau)2K}\langle\sigma_{2,1}(0)\sigma_{1,2}(\tau)\rangle + \int_\tau^t dt' e^{(t'-t)2K}\Gamma^* e^{-2K(t'-\tau)} \\ &= e^{-(t-\tau)2K}\langle\sigma_{2,1}(0)\sigma_{1,2}(\tau)\rangle + \Gamma^* e^{-2K(t-\tau)}(t-\tau) \ . \end{aligned} \tag{4.49}$$

Using the solution (4.47) this yields

$$\langle\sigma_{2,1}(t-\tau)\sigma_{1,2}(t)\rangle = e^{K\tau-2Kt} + \Gamma^* e^{-2K(t-\tau)}(t-\tau)\ . \tag{4.50}$$

We therefore conclude the differential equation

$$\begin{aligned}\frac{d}{dt}\langle\sigma_{2,2}(t)\rangle &= -2K\langle\sigma_{2,2}(t)\rangle + 2Re\left(-KRe^{-i2k_0d}e^{K\tau-2Kt} + K^2R^2(t-\tau)e^{-2K(t-\tau)}\right)\\ &= -2K\langle\sigma_{2,2}(t)\rangle + 2\left(-KR\cos(2k_0d)e^{K\tau-2Kt} + K^2R^2(t-\tau)e^{-2K(t-\tau)}\right)\end{aligned} \tag{4.51}$$

We will now again use Duhamels formula and obtain

$$\begin{aligned}\langle\sigma_{2,2}(t)\rangle &= e^{-(t-\tau)2K}\langle\sigma_{2,2}(\tau)\rangle - 2KR\cos(2k_0d)\int_\tau^t dt' e^{(t'-t)2K}e^{K\tau-2Kt'}\\ &\quad + 2K^2R^2\int_\tau^t dt' e^{(t'-t)2K}(t'-\tau)e^{-2K(t'-\tau)}\\ &= e^{-(t-\tau)2K}e^{-2K\tau} - 2KR\cos(2k_0d)\int_\tau^t dt' e^{K(\tau-2t)}\\ &\quad + 2K^2R^2e^{-2K(t-\tau)}\int_\tau^t dt'(t'-\tau)\\ &= e^{-2Kt}\left(1-2KR\cos(2k_0d)(t-\tau)e^{K\tau}\right) + K^2R^2e^{-2K(t-\tau)}(t-\tau)^2\ .\end{aligned} \tag{4.52}$$

Under the assumption that $K\tau \gg 1$ the approximation $e^{-2K\tau} \approx 0$ holds. Hence

$$\langle\sigma_{2,2}(t)\rangle = |\Gamma|^2 e^{-2K(t-\tau)}(t-\tau)^2\ . \tag{4.53}$$

The following graph shows the numerical simulation of the solution (4.53) on the interval $[0, 2\tau)$.

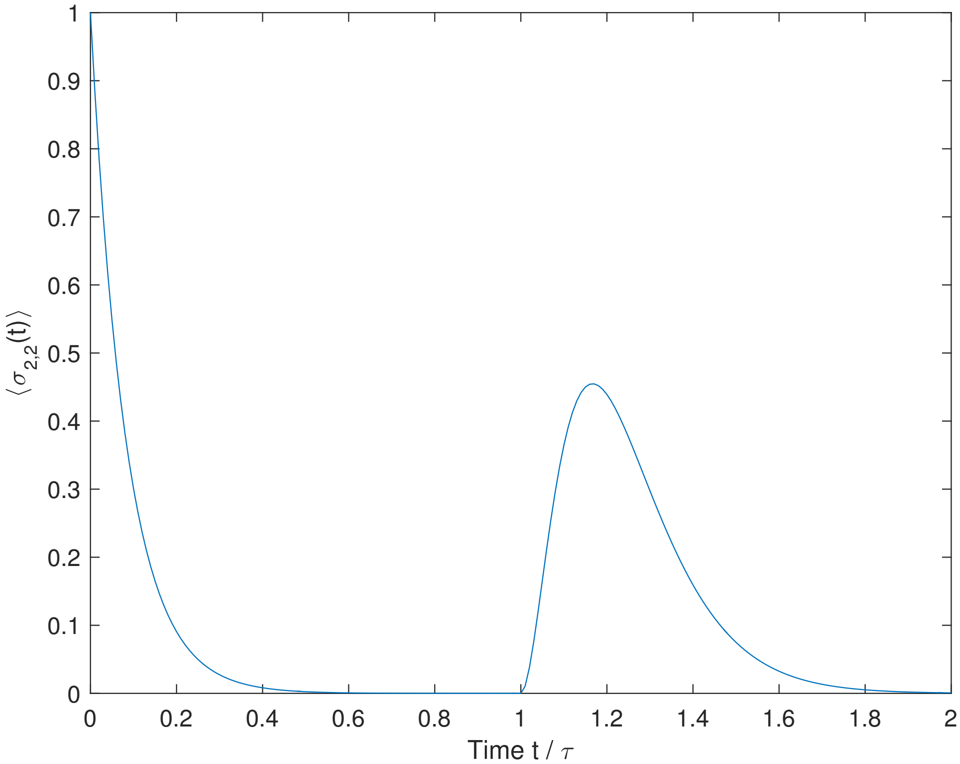

Figure 2: Numerical simulation of the solution (4.53) on the interval $[0, 2\tau)$.

5 Conclusion

This thesis presented a derivation for the differential equations describing the probability amplitudes of he state functions using Schrödingers picture. Here an orthogonal transformation into the rotating frame with respect to the transition frequency of the initially excited atom ω_0 was used. The mathematical analysis of this transformation was done by using the Baked-Campbell-Hausdorff expansion. This led to a system of linear differential equations. Following a detailed analysis of the work *Quantum theory of an atom near partially reflecting walls* by R. J. Cook and P. W. Milonni [2] was presented. It led to a delay differential equation with an initial value. Due to the special structure of this delay differential equation an analytical solution was possible. A description of that solution with its prove terminates the section, where Schrödingers picture is the considered formulation of quantum mechanics.

The transformation into Heisenbergs picture led to similar calculations on the mathematical level of operators which yielded also a delay differential operator equation. With special physical constrains this equation was transferred to an operator analogon to the delay differential equation in Schrödingers picture. This delay differential operator equation has no initial value as the physical meaning of an operator itself is unclear. The expected value as result of an experimental measurement however has an initial value. Therefore the expected value of the delay differential operator equation was solvable. The solution of operator describing the excited state of the initially excited atom was then presented and derived. In the end a numerical simulation of that calculated solution was given.

This thesis laid the foundation for further analysis of the quantum theory of an atom near partially reflecting walls using Heisenbergs picture. It gave an alternative approach to the describing delay differential operator equation which was till now obtained by using the phenomenological motivated interacting Hamiltonian

$$H_I = i\hbar \sum_{j=0}^{N} \sum_{n\in\mathbb{N}} C_n \sin(nL) \left(\sigma_{2,1}^{(j)} c_n e^{ik_n \cdot x_j} - \sigma_{1,2}^{(j)} c_n^\dagger e^{-ik_n \cdot x_j} \right) \ . \tag{5.1}$$

Hence a mathematical analysis in terms of a transformation between the two Hamiltonians is of interest. Further this thesis only discusses the one excitation limit. But the presented calculations can be used to consider an excited or partially excited dielectric medium. This will lead to the physical analysis of time delayed feedback of excited media. Moreover the for Heisenbergs picture characteristic and particularly difficult noise term was omitted due to the one excitation limit which led to the normal form of operator products. Hence the presented calculations also lead to a description of the system using quantum noise.

6 Appendix

In this section we will present the calculations used in this thesis.

6.1 Description of Atom-Field Interaction Using Schrödingers Picture

6.1.1 Unitary Transformation of Hamilton Defining Operators

For the transformation of the interaction Hamiltonian the Baker-Campbell-Hausdorff expansion had to be applied several times to different operators. We here present the detailed calculations for the Hamiltonian defining operator which were not explicitly presented in the thesis. For the operator $\sigma_{1,2}^{(j)}$ we find

$$\begin{aligned}\left[\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\ ,\ \sigma_{1,2}^{(J)}\right] &= \sum_{j=0}^{N}\left[\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\ ,\ \sigma_{1,2}^{(J)}\right] = \sum_{j=0}^{N}\delta_{j,J}\left[\sigma_{2,1}^{(J)}\sigma_{1,2}^{(J)}\ ,\ \sigma_{1,2}^{(J)}\right] \\ &= \left[\sigma_{2,1}^{(J)}\ ,\ \sigma_{1,2}^{(J)}\right]\sigma_{1,2}^{(J)} = \left(\sigma_{2,2}^{(J)} - \sigma_{1,1}^{(J)}\right)\sigma_{1,2}^{(J)} = -\sigma_{1,2}^{(J)}\ .\end{aligned} \tag{6.1}$$

We conclude that

$$\left[\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\ ,\ \sigma_{1,2}^{(J)}\right]_m = (-1)^m\sigma_{1,2}^{(J)} \tag{6.2}$$

which yields

$$\begin{aligned}&exp\left(i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t\right)\sigma_{1,2}^{(J)}exp\left(-i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t\right) \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\left[i\omega_0\left(\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\right)t\ ,\ \sigma_{1,2}^{(J)}\right]_m = \sum_{m=0}^{\infty}\frac{(i\omega_0 t)^m}{m!}\left[\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}\ ,\ \sigma_{1,2}^{(J)}\right]_m \\ &= \sum_{m=0}^{\infty}\frac{(i\omega_0 t)^m}{m!}(-1)^m\sigma_{1,2}^{(J)} = e^{-i\omega_0 t}\sigma_{1,2}^{(J)}\ .\end{aligned} \tag{6.3}$$

We now proceed in the same way for the operator c_l. The Baker-Campbell-Hausdorff expansion yields

$$exp\left(i\omega_0\sum_{n\in\mathbb{N}}c_n^\dagger c_n t\right)c_l exp\left(-i\omega_0\sum_{n\in\mathbb{N}}c_n^\dagger c_n t\right) = \sum_{m=0}^{\infty}\frac{1}{m!}\left[i\omega_0\left(\sum_{n\in\mathbb{N}} where c_n^\dagger c_n\right)t\ ,\ c_l\right]_m\ . \tag{6.4}$$

We start again with the Lie bracket which is equivalent to $[\cdot,\cdot]_1$ and find

$$\left[\sum_{n\in\mathbb{N}}c_n^\dagger c_n\ ,\ c_l\right] = \sum_{n\in\mathbb{N}}\delta_{n,l}\left[c_n^\dagger c_n\ ,\ c_l\right] = \left[c_l^\dagger\ ,\ c_l\right]c_l = -c_l\ . \tag{6.5}$$

We conclude by the linearity of the Lie bracket that

$$\left[\sum_{n\in\mathbb{N}} c_n^\dagger c_n \ , \ c_l\right]_m = (-1)^m c_l \ . \tag{6.6}$$

Hence

$$\begin{aligned} exp\left(i\omega_0 \sum_{n\in\mathbb{N}} c_n^\dagger c_n t\right) c_l exp\left(-i\omega_0 \sum_{n\in\mathbb{N}} c_n^\dagger c_n t\right) &= \sum_{m=0}^{\infty} \frac{1}{m!} \left[i\omega_0 \left(\sum_{n\in\mathbb{N}} c_n^\dagger c_n\right) t \ , \ c_l\right]_m \\ = \sum_{m=0}^{\infty} \frac{(it\omega_0)^m}{m!} \left[\sum_{n\in\mathbb{N}} c_n^\dagger c_n \ , \ c_l\right]_m &= \sum_{m=0}^{\infty} \frac{(-it\omega_0)^m}{m!} c_l = e^{-i\omega_l t} c_l \ . \end{aligned} \tag{6.7}$$

For the complex conjugate we find

$$\left[\sum_{n\in\mathbb{N}} c_n^\dagger c_n \ , \ c_l^\dagger\right] = \sum_{n\in\mathbb{N}} \delta_{n,l} \left[c_n^\dagger c_n \ , \ c_l^\dagger\right] = c_l^\dagger \left[c_l \ , \ c_l^\dagger\right] = c_l^\dagger \ . \tag{6.8}$$

Again by linearity of the Lie bracket we conclude that

$$\left[\sum_{n\in\mathbb{N}} c_n^\dagger c_n \ , \ c_l^\dagger\right]_m = c_l^\dagger \ . \tag{6.9}$$

Hence

$$\begin{aligned} exp\left(i\omega_0 \sum_{n\in\mathbb{N}} c_n^\dagger c_n t\right) c_l^\dagger exp\left(-i\omega_0 \sum_{n\in\mathbb{N}} c_n^\dagger c_n t\right) &= \sum_{m=0}^{\infty} \frac{1}{m!} \left[i\omega_0 \left(\sum_{n\in\mathbb{N}} c_n^\dagger c_n\right) t \ , \ c_l^\dagger\right]_m \\ = \sum_{m=0}^{\infty} \frac{(it\omega_0)^m}{m!} \left[\sum_{n\in\mathbb{N}} c_n^\dagger c_n \ , \ c_l^\dagger\right]_m &= \sum_{m=0}^{\infty} \frac{(it\omega_0)^m}{m!} c_l^\dagger = e^{i\omega_l t} c_l^\dagger \ . \end{aligned} \tag{6.10}$$

6.1.2 Unitary Transformation of Describing Hamiltonian

The Baker-Campbell-Hausdorff expansion mad it possible to calculate the transformation of operators defining the Hamiltonian. These transformations now will become of use for the Transformation of the interaction Hamiltonian. We will insert the definition of the unitary operator T. Further we will use that the flip operators and the photon operators commute. Then the transformation of operators defining the Hamiltonian will become applicable. We find

$$
\begin{aligned}
&TH_IT^* \\
&= Ti\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l\left(\sigma_{2,1}^{(J)}c_l e^{ik_l\cdot x_J} - \sigma_{1,2}^{(J)}c_l^{\dagger}e^{-ik_l\cdot x_J}\right)T^* \\
&= i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l e^{ik_l\cdot x_J}T\sigma_{2,1}^{(J)}c_lT^* - i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l e^{-ik_l\cdot x_J}T\sigma_{1,2}^{(J)}c_l^{\dagger}T^* \\
&= i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l e^{ik_l\cdot x_J}e^{i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t}e^{i\omega_0\sum_{n\in\mathbb{N}}c_n^{\dagger}c_nt}\sigma_{2,1}^{(J)}c_l e^{-i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t}e^{-i\omega_0\sum_{n\in\mathbb{N}}c_n^{\dagger}c_nt} \\
&\quad - i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l e^{-ik_l\cdot x_J}e^{i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t}e^{i\omega_0\sum_{n\in\mathbb{N}}c_n^{\dagger}c_nt}\sigma_{1,2}^{(J)}c_l^{\dagger}e^{-i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t}e^{-i\omega_0\sum_{n\in\mathbb{N}}c_n^{\dagger}c_nt} \\
&= i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l e^{ik_l\cdot x_J}e^{i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t}\sigma_{2,1}^{(J)}e^{-i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t}e^{i\omega_0\sum_{n\in\mathbb{N}}c_n^{\dagger}c_nt}c_l e^{-i\omega_0\sum_{n\in\mathbb{N}}c_n^{\dagger}c_nt} \\
&\quad - i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l e^{-ik_l\cdot x_J}e^{i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t}\sigma_{1,2}^{(J)}e^{-i\omega_0\sum_{j=0}^{N}\sigma_{2,1}^{(j)}\sigma_{1,2}^{(j)}t}e^{i\omega_0\sum_{n\in\mathbb{N}}c_n^{\dagger}c_nt}c_l^{\dagger}e^{-i\omega_0\sum_{n\in\mathbb{N}}c_n^{\dagger}c_nt} \\
&= i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l e^{ik_l\cdot x_J}e^{i\omega_0t}\sigma_{2,1}^{(J)}e^{-i\omega_0t}c_l - i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l e^{-ik_l\cdot x_J}e^{-i\omega_0t}\sigma_{1,2}^{(J)}e^{i\omega_0t}c_l^{\dagger} \\
&= i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l e^{ik_l\cdot x_J}\sigma_{2,1}^{(J)}c_l - i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l e^{-ik_l\cdot x_J}\sigma_{1,2}^{(J)}c_l^{\dagger} \\
&= i\hbar\sum_{J=0}^{N}\sum_{l\in\mathbb{N}} C_l\left(e^{ik_l\cdot x_J}\sigma_{2,1}^{(J)}c_l - e^{-ik_l\cdot x_J}\sigma_{1,2}^{(J)}c_l^{\dagger}\right) .
\end{aligned}
\tag{6.11}
$$

6.1.3 Applying Schrödingers Equation to the Transformed Hamiltonian

We here present the calculations using Schrödingers equation to deduce the final differential equations describing the probability amplitudes that are the foundation of the work *Quantum theory of an atom near partially reflecting walls* by R. J. Cook and P. W. Milonni [2]. We start by calculating the right side of Schrödingers equation. We find

$$\begin{aligned} &i\hbar \sum_{J=0}^{N} \sum_{l\in\mathbb{N}} C_l e^{ik_l\cdot x_J} \sigma_{2,1}^{(J)} c_l |\Psi\rangle \\ &= i\hbar \sum_{J=1}^{N} \sum_{l\in\mathbb{N}} C_l e^{ik_l\cdot x_J} a_l(t) |1,0,\{1,\ldots 1, \underbrace{2}_{\text{J-th atom}}, 1\ldots,1\}\rangle + i\hbar \sum_{l\in\mathbb{N}} C_l e^{ik_l\cdot x_0} a_l(t) |2,0,\{1,\ldots,1\}\rangle \\ &= i\hbar \sum_{l\in\mathbb{N}} C_l \; a_l(t) \left(e^{ik_l\cdot x_0} |2,0,\{1,\ldots,1\}\rangle + \sum_{J=1}^{N} e^{ik_l\cdot x_J} |1,0,\{1,\ldots 1, \underbrace{2}_{\text{J-th atom}}, 1\ldots,1\}\rangle \right) \; . \end{aligned} \tag{6.12}$$

We further see that for all $J \in \{0,...,N\}$ the operator $\sigma_{1,2}^{(J)}$ is only well-defined on the states $|2,0,\{1...1\}\rangle$ and $|1,0,\{1... \underbrace{1\;2\;1}_{\text{J-th atom}} ...1\}\rangle$. Hence

$$\begin{aligned} &i\hbar \sum_{J=0}^{N} \sum_{l\in\mathbb{N}} C_l e^{-ik_l\cdot x_J} \sigma_{1,2}^{(J)} c_l^\dagger |\Psi\rangle \\ &= i\hbar \sum_{l\in\mathbb{N}} C_l \left(e^{-ik_l\cdot x_0} \; b(t) + \sum_{J=1}^{N} e^{-ik_l\cdot x_J} \; b_J(t) \right) |1,l,\{1,\ldots,1\}\rangle \; . \end{aligned} \tag{6.13}$$

Further we obtain

$$\hbar \sum_{J=1}^{N} (\omega-\omega_0) \sigma_{2,1}^{(J)} \sigma_{1,2}^{(J)} |\psi\rangle = \hbar \sum_{J=1}^{N} (\omega-\omega_0) b_J(t) |1,0,\{1,\ldots 1, \underbrace{2}_{\text{J-th atom}}, 1\ldots,1\}\rangle \tag{6.14}$$

and

$$\hbar \sum_{l\in\mathbb{N}} (\omega_l-\omega_0) c_l^\dagger c_l |\psi\rangle = \hbar \sum_{l\in\mathbb{N}} (\omega_l-\omega_0) a_l(t) |1,l,\{1,\ldots,1\}\rangle \; . \tag{6.15}$$

We therefore conclude that

$$\begin{aligned} H|\Psi\rangle = & i\hbar \sum_{l\in\mathbb{N}} C_l e^{ik_l\cdot x_0} a_l(t) |2,0,\{1...1\}\rangle \\ & + i\hbar \sum_{J=1}^{N} \sum_{l\in\mathbb{N}} C_l e^{ik_l\cdot x_J} a_l(t) |1,0,\{1...\underbrace{1\ 2\ 1}_{\text{J-th atom}}...1\}\rangle \\ & - i\hbar \sum_{l\in\mathbb{N}} C_l \left(e^{-ik_l\cdot x_0}\ b(t) + \sum_{J=1}^{N} e^{-ik_l\cdot x_J}\ b_J(t) \right) |1,l,\{1,...,1\}\rangle \\ & + \hbar \sum_{J=1}^{N} (\omega-\omega_0) b_J(t) |1,0,\{1,...1,\underbrace{\ 2\ }_{\text{J-th atom}},1...,1\}\rangle \\ & + \hbar \sum_{l\in\mathbb{N}} (\omega_l-\omega_0) a_l(t) |1,l,\{1,...,1\}\rangle \end{aligned} \tag{6.16}$$

Schrödingers equation multiplied from the right by the dual operators corresponding to the basis functions yields

$$\begin{aligned} \dot{b}(t) &= -\sum_{n\in\mathbb{N}} C_n a_n(t) e^{ik_n\cdot x_0} \\ \dot{b}_j(t) &= -(\omega-\omega_0) b_j(t) - \sum_{n\in\mathbb{N}} C_n a_n(t) e^{ik_n\cdot x_j} & ,\forall j\in\{1,...,N\} \\ \dot{a}_n(t) &= -(\omega_n-\omega_0) a_n(t) + C_n b(t) e^{-ik_n\cdot x_0} + C_n \sum_{j=1}^{N} b_j(t) e^{-ik_n\cdot x_j} & ,\forall n\in\mathbb{N} \end{aligned} \tag{6.17}$$

6.1.4 Analytical Solution of a_n

We will present the detailed calculations for the solution of a_n. We will start by checking the applicability of Duhamels formula to the third equation of the system (3.26). We therefore define the following auxiliary functions

$$\begin{aligned} e(t) &:= C_n b(t) e^{-ik_n \cdot x_0} + C_n \sum_{j=1}^{N} b_j(t) e^{-ik_n \cdot x_j} \\ d &:= i(\omega_n - \omega_0) \ . \end{aligned} \tag{6.18}$$

Using these functions the third differential equation of formula (3.26) can be expressed as

$$\begin{cases} \dot{a}_n(t) + d\ a_n(t) = e(t) \\ \qquad\qquad a_n(0) = 0 \ . \end{cases} \tag{6.19}$$

It is obvious that d as a constant function is an element of $\mathcal{C}(\mathbb{R}, L(\mathbb{R}))$. Under the assumption that b and b_j are in $\mathcal{C}(\mathbb{R}, \mathbb{R})$ the considered function e is in $\mathcal{C}(\mathbb{R}, \mathbb{R})$. In (6.19) we used $I = X = \mathbb{R}$ and therefore $0 \in I = \mathbb{R}$. Hence Duhamels formula is applicable to the differential equation (6.19). We obtain

$$\begin{aligned} a_n(t) &= \int_0^t dt' e^{(t'-t)d} e(t') \\ &= \int_0^t dt' e^{i(t'-t)(\omega_n - \omega_0)} \left(C_n b(t') e^{-ik_n \cdot x_0} + C_n \sum_{j=1}^{N} b_j(t') e^{-ik_n \cdot x_j} \right) \ . \end{aligned} \tag{6.20}$$

6.1.5 Differential Equations with Eliminated Electromagnetic Field

The calculated solution of $a_n(t)$ is used to reduce the system of differential equations. To eliminate the dependency of $b(t)$ and $b_j(t)$ with $j \in \{1, ..., N\}$ of $a_n(t)$ we insert this solution of (6.19) into the first two equations in (3.26). This will lead to the desired reduction of considered equations. For $b(t)$ we obtain

$$\begin{aligned} \dot{b}(t) &= -\sum_{n\in\mathbb{N}} C_n \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} \left(C_n b(t') e^{-ik_n\cdot x_0} + C_n \sum_{j=1}^N b_j(t') e^{-ik_n\cdot x_j} \right) e^{ik\cdot x_0} \\ &= -\sum_{n\in\mathbb{N}} C_n^2 \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b(t') e^{ix_0\cdot(k-k_n)} \\ &\quad - \sum_{n\in\mathbb{N}} \sum_{j=1}^N C_n^2 \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b_j(t') e^{i(k\cdot x_0 - k_n\cdot x_j)} \ . \end{aligned} \tag{6.21}$$

We will now restrict our system to modes of the electromagnetic field with wave vectors k_n parallel to the z-axis. Thus we can take

$$C_n = \frac{\mu}{\hbar} \left(\frac{2\pi\hbar\omega_n}{AL} \right)^{\frac{1}{2}} \ , \tag{6.22}$$

where A is an effective area, L is the length along the z-axis of the quantised box and μ is the magnitude of the transition dipole moment of each two-state atom. Hence we can rewrite (6.21) as

$$\dot{b}(t) = -\sum_{n\in\mathbb{N}} C_n^2 \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b(t') - \sum_{n\in\mathbb{N}} \sum_{j=1}^N C_n^2 \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b_j(t') e^{ik_n(z_0-z_j)} \ . \tag{6.23}$$

In the same way we will now insert (6.20) into the second equation of (6.19) and obtain

$$\begin{aligned} \dot{b}_j(t) &= -(\omega-\omega_j) b_j(t) \\ &\quad - \sum_{n\in\mathbb{N}} C_n e^{ik_n\cdot x_j} \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} \left(C_n b(t') e^{-ik_n\cdot x_0} + C_n \sum_{l=1}^N b_l(t') e^{-ik_n\cdot x_l} \right) \\ &= -(\omega-\omega_j) b_j(t) - \sum_{n\in\mathbb{N}} C_n^2 e^{ik_n(z_j-z_0)} \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b(t') \\ &\quad - \sum_{n\in\mathbb{N}} \sum_{l=1}^N C_n^2 e^{ik_n(z_j-z_l)} \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b_l(t') \ . \end{aligned} \tag{6.24}$$

6.1.6 Length Limit of Quantised Box

We will now pass to the limit in which the length L of the quantisation box goes to infinity. We first analyse the term

$$
\begin{aligned}
\sum_{n\in\mathbb{N}} C_n^2 e^{i(\omega_n-\omega_0)(t'-t)} &= \frac{2\pi\mu^2}{\hbar AL}\sum_{n\in\mathbb{N}}\omega_n e^{i(\omega_n-\omega_0)(t'-t)} = \frac{2\pi\mu^2}{\hbar AL}\sum_{n\in\mathbb{N}}\frac{\Delta k}{\Delta k}\omega_n e^{i(\omega_n-\omega_0)(t'-t)} \\
&= \frac{2\pi\mu^2}{\hbar AL}\frac{L}{2\pi}\sum_{n\in\mathbb{N}}\Delta k\omega_n e^{i(\omega_n-\omega_0)(t'-t)} \longrightarrow \frac{\mu^2}{\hbar A}\int_{-\infty}^{\infty} dk'\omega' e^{i(\omega'-\omega_0)(t'-t)} \\
&= \frac{\mu^2}{\hbar Ac}\int_{-\infty}^{\infty} dk' c(ck') e^{i(ck')-\omega_0)(t'-t)} = \frac{\mu^2}{\hbar Ac}\int_{-\infty}^{\infty} d\omega'\omega' e^{i(\omega'-\omega_0)(t'-t)} \\
&= \frac{\mu^2}{\hbar Ac}e^{-i\omega_0(t'-t)}\int_{-\infty}^{\infty} d\omega'\omega' e^{i\omega'(t'-t)} = \frac{\mu^2}{\hbar Ac}e^{-i\omega_0(t'-t)}\int_{-\infty}^{\infty} d\omega'\frac{d}{dt'}(-i)e^{i\omega'(t'-t)} \\
&= -\frac{i\mu^2}{\hbar Ac}e^{-i\omega_0(t'-t)}\frac{d}{dt'}\int_{-\infty}^{\infty} d\omega' e^{i\omega'(t'-t)} = -\frac{2\pi i\mu^2}{\hbar Ac}e^{-i\omega_0(t'-t)}\frac{d}{dt'}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{i\omega'(t'-t)}
\end{aligned}
\tag{6.25}
$$

We will now apply this result to the first term of (6.23). Here we get

$$
\begin{aligned}
&\sum_{n\in\mathbb{N}} C_n^2\int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)}b(t') = \int_0^t dt' b(t')\sum_{n\in\mathbb{N}} C_n^2 e^{i(t'-t)(\omega_n-\omega_0)} \\
&\longrightarrow -\frac{2\pi i\mu^2}{\hbar Ac}\int_0^t dt' b(t') e^{-i\omega_0(t'-t)}\frac{d}{dt'}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{i\omega'(t'-t)} \\
&= -\frac{2\pi i\mu^2}{\hbar Ac}\left(\left(b(t')e^{-i\omega_0(t'-t)}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{i\omega'(t'-t)}}_{=\delta(t'-t)}\right)\Bigg|_{t'=0}^{t'=t}\right. \\
&\qquad \left. - \int_0^t dt'\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{i\omega'(t'-t)}}_{=\delta(t'-t)}\frac{d}{dt'}b(t')e^{-i\omega_0(t'-t)}\right) \\
&= -\frac{2\pi i\mu^2}{\hbar Ac}\left(b(t)(-t) - \frac{1}{2}\left\langle\delta, \frac{d}{dt'}b(t'+t)e^{-i\omega_0 t'}\right\rangle\right) ,
\end{aligned}
\tag{6.26}
$$

where we have used the result

$$
\int_0^t dt' f(t')\delta\circ g^{-1}(t') = \frac{1}{2}\langle\delta, f\circ g\rangle = \frac{1}{2}f\circ g(0) ,
$$

which can be seen in [7]. Here $g:\mathbb{R}\to\mathbb{R}; t'\mapsto t'+t$ is bijective and the general property of the delta distribution

$$
\delta(f) = \langle\delta, f\rangle = f(0) \tag{6.27}
$$

holds. We will now take a closer look to the derivative used in the duality pairing. Here we obtain

$$\frac{d}{dt'}b(t')e^{-i\omega_0(t'-t)} = \dot{b}(t')e^{-i\omega_0(t'-t)} - i\omega_0 b(t')e^{-i\omega_0(t'-t)} \ . \tag{6.28}$$

It is important to notice that in the here discussed case $b(t)$ varies much slower than $e^{-i\omega_0 t}$. This implies that

$$\frac{d}{dt'}b(t')e^{-i\omega_0(t'-t)} \approx -i\omega_0 b(t')e^{-i\omega_0(t'-t)} \ . \tag{6.29}$$

In the following we will use this approximation and we refrain from using the approximative-equal-to-sign. Instead we set

$$\frac{d}{dt'}b(t')e^{-i\omega_0(t'-t)} = -i\omega_0 b(t')e^{-i\omega_0(t'-t)} \ . \tag{6.30}$$

This leads to

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} C_n^2 \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b(t') \longrightarrow -\frac{2\pi i\mu^2}{\hbar Ac}\left(b(t)(-t) - \frac{1}{2}\left\langle \delta, \frac{d}{dt'}b(t'+t)e^{-i\omega_0 t'}\right\rangle\right) \\
&= -\frac{2\pi i\mu^2}{\hbar Ac}\left(b(t)(-t) - \frac{1}{2}\left\langle \delta, -i\omega_0 b(t'+t)e^{-i\omega_0 t'}\right\rangle\right) \\
&= -\frac{2\pi i\mu^2}{\hbar Ac}\left(b(t)(-t) + i\omega_0 b(t)\right) = \frac{2\pi i\mu^2}{\hbar Ac}b(t)t + \frac{\pi\omega_0\mu^2}{\hbar Ac}b(t) \ .
\end{aligned} \tag{6.31}$$

We will furthermore ignore the term $b(t)t2\pi i\mu^2/(\hbar Ac)$ which corresponds to a single-atom frequency shift. This yields

$$\sum_{n\in\mathbb{N}} C_n^2 \int_0^t dt' e^{i(t'-t)(\omega_n-\omega_0)} b(t') \longrightarrow \frac{\pi\mu^2\omega_0}{\hbar Ac}b(t) \ . \tag{6.32}$$

We will now analyse the second term from (6.23). Analogously to the first term we start with

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} C_n^2 e^{ik_n Z} e^{i(\omega_n-\omega_0)(t'-t)} = \frac{2\pi\mu^2}{\hbar AL}\sum_{n\in\mathbb{N}} \omega_n e^{ik_n Z} e^{i(\omega_n-\omega_0)(t'-t)} \\
&\quad = \frac{2\pi\mu^2}{\hbar AL}\frac{L}{2\pi}\sum_{n\in\mathbb{N}} \Delta k\omega_n e^{ik_n Z} e^{i(\omega_n-\omega_0)(t'-t)} \longrightarrow \frac{\mu^2}{\hbar A}\int_{-\infty}^{\infty} dk'\omega' e^{ik'Z} e^{i(\omega'-\omega_0)(t'-t)} \\
&\quad = \frac{\mu^2}{\hbar A}\left(\int_{-\infty}^{\infty} dk'\omega' \cos(k'Z) e^{i(\omega'-\omega_0)(t'-t)} + i\int_{-\infty}^{\infty} dk'\omega' \sin(k'Z) e^{i(\omega'-\omega_0)(t'-t)}\right) \ .
\end{aligned} \tag{6.33}$$

It is known that

$$f : \mathbb{R} \to \mathbb{R}; k' \mapsto \omega' \sin(k'Z) \tag{6.34}$$

is an uneven function. Eulers formula yields that

$$g : \mathbb{R} \to \mathbb{R}; k' \mapsto e^{i(\omega'-\omega_0)(t'-t)} = \cos((\omega'-\omega_0)(t'-t)) + i\sin((\omega'-\omega_0)(t'-t)) \tag{6.35}$$

is an addition of an even function and an uneven function. As the product of an even function and an uneven function is an uneven function just as the product of an uneven function and an uneven function we conclude that

$$\int_{-\infty}^{\infty} dk'\omega' \sin(k'Z)e^{i(\omega'-\omega_0)(t'-t)} = 0 \ . \tag{6.36}$$

This yields

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} C_n^2 e^{ik_nZ} e^{i(\omega_n-\omega_0)(t'-t)} \\
&\quad \longrightarrow \frac{\mu^2}{\hbar A}\int_{-\infty}^{\infty} dk'\omega' \cos(k'Z)e^{i(\omega'-\omega_0)(t'-t)} = \frac{\mu^2}{\hbar Ac}\int_{-\infty}^{\infty} d\omega'\omega' \cos(\omega' Z/c)e^{i(\omega'-\omega_0)(t'-t)} \\
&\quad = \frac{\mu^2}{2\hbar Ac}\int_{-\infty}^{\infty} d\omega'\omega' \left(e^{i\omega' Z/c} + e^{-i\omega' Z/c}\right) e^{i(\omega'-\omega_0)(t'-t)} \\
&\quad = \frac{\mu^2}{2\hbar Ac}\int_{-\infty}^{\infty} d\omega'\omega' \left(e^{i\omega'(t'-t+Z/c)} + e^{i\omega'(t'-t-Z/c)}\right) e^{-i\omega_0(t'-t)} \\
&\quad = \frac{\mu^2}{2\hbar Ac} e^{-i\omega_0(t'-t)} \int_{-\infty}^{\infty} d\omega' \frac{d}{dt'}(-i) \left(e^{i\omega'(t'-t+Z/c)} + e^{i\omega'(t'-t-Z/c)}\right) \\
&\quad = -\frac{i\pi\mu^2}{\hbar Ac} e^{-i\omega_0(t'-t)} \frac{d}{dt'}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{i\omega'(t'-t+Z/c)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{i\omega'(t'-t-Z/c)}\right) \ .
\end{aligned} \tag{6.37}$$

This is used to rewrite the second term in (6.23) as follows

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} C_n^2 e^{ik_n(z_0-z_j)} \int_0^t dt' b_j(t') e^{i(\omega_n-\omega_0)(t'-t)} = \int_0^t dt' b_j(t') \sum_{n\in\mathbb{N}} C_n^2 e^{ik_n(z_0-z_j)} e^{i(\omega_n-\omega_0)(t'-t)} \\
&\longrightarrow \int_0^t dt' b_j(t') \left(-\frac{i\pi\mu^2}{\hbar Ac} e^{-i\omega_0(t'-t)} \frac{d}{dt'}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{i\omega'(t'-t+(z_0-z_j)/c)}\right.\right. \\
&\qquad \left.\left. + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' e^{i\omega'(t'-t-(z_0-z_j)/c)}\right)\right) \\
&= -\frac{i\pi\mu^2}{\hbar Ac}\left(b_j(t') e^{-i\omega_0(t'-t)} (\delta(t'-t+(z_0-z_j)/c) + \delta(t'-t-(z_0-z_j)/c))\Big|_{t'=0}^{t}\right. \\
&\qquad \left. - \int_0^t dt' (\delta(t'-t+(z_0-z_j)/c) + \delta(t'-t-(z_0-z_j)/c)) \frac{d}{dt'}\left(b_j(t') e^{-i\omega_0(t'-t)}\right)\right) \\
&= \frac{i\pi\mu^2}{\hbar Ac} 2b_j(t)t + \frac{i\pi\mu^2}{\hbar Ac}\int_{-t+(z_0-z_j)/c}^{(z_0-z_j)/c} dt' \delta(t') \frac{d}{dt'}\left(b_j(t'+t-(z_0-z_j)/c) e^{-i\omega_0(t'-(z_0-z_j)/c)}\right) \\
&\quad + \frac{i\pi\mu^2}{\hbar Ac}\int_{-t-(z_0-z_j)/c}^{-(z_0-z_j)/c} dt' \delta(t') \frac{d}{dt'}\left(b_j(t'+t+(z_0-z_j)/c) e^{-i\omega_0(t'+(z_0-z_j)/c)}\right) \\
&= \frac{i\pi\mu^2}{\hbar Ac} 2b_j(t)t + \frac{i\pi\mu^2}{\hbar Ac}\langle \delta \ , \ g_+ \rangle + \frac{i\pi\mu^2}{\hbar Ac}\langle \delta \ , \ g_- \rangle
\end{aligned} \tag{6.38}$$

with

$$g_{\pm}(t') = \Theta(\pm(z_0 - z_j)/c - t')\Theta(t' + t \mp (z_0 - z_j)/c))\frac{d}{ds} \times \Big(b_j(s + t \mp (z_0 - z_j)/c)e^{-i\omega_0(s \mp (z_0 - z_j)/c)}\Big)\Big|_{s=t'} \tag{6.39}$$

and Θ the Heaviside step function.
The analysis the derivatives in the duality pairings yields

$$\frac{d}{dt'}\Big(b_j(t' + t + (z_0 - z_j)/c)e^{-i\omega_0(t' + (z_0 - z_j)/c)}\Big) = -i\omega_0 b_j(t' + t + (z_0 - z_j)/c)e^{-i\omega_0(t' + (z_0 - z_j)/c)} \tag{6.40}$$

and

$$\frac{d}{dt'}\Big(b_j(t' + t - (z_0 - z_j)/c)e^{-i\omega_0(t' - (z_0 - z_j)/c)}\Big) = -i\omega_0 b_j(t' + t - (z_0 - z_j)/c)e^{-i\omega_0(t' - (z_0 - z_j)/c)} \ , \tag{6.41}$$

where we have used the same approximation as in (6.29). We can therefore rewrite the duality pairings in (6.38) as

$$\begin{aligned} &\frac{i\pi\mu^2}{\hbar Ac}\langle\delta\ ,\ g_+\rangle \\ =&\frac{\pi\mu^2\omega_0}{\hbar Ac}\int_{\infty}^{\infty} dt'\delta(t')\Theta((z_0 - z_j)/c - t')\Theta(t' + t - (z_0 - z_j)/c)) \\ &\times b_j(t' + t - (z_0 - z_j)/c)e^{-i\omega_0(t' - (z_0 - z_j)/c)} \\ =&\frac{\pi\mu^2\omega_0}{\hbar Ac}\Theta((z_0 - z_j)/c)\Theta(t - (z_0 - z_j)/c))b_j(t - (z_0 - z_j)/c)e^{i\omega_0(z_0 - z_j)/c} \\ =&\frac{\pi\mu^2\omega_0}{2\hbar Ac}b_j(t - |z_0 - z_j|/c)e^{i\omega_0|z_0 - z_j|/c}\Theta(t - |z_0 - z_j|/c)) \ . \end{aligned} \tag{6.42}$$

Analogously we find

$$\frac{i\pi\mu^2}{\hbar Ac}\langle\delta\ ,\ g_-\rangle = \frac{\omega_0\pi\mu^2}{2\hbar Ac}b_j(t - |z_0 - z_j|/c)e^{i\omega_0|z_0 - z_j|/c}\Theta(t - |z_0 - z_j|/c) \ . \tag{6.43}$$

Hence it follows

$$\sum_{n\in\mathbb{N}} C_n^2 e^{ik_n(z_0 - z_j)}\int_0^t dt' b_j(t')e^{i(\omega_n - \omega_0)(t' - t)} \longrightarrow \frac{\omega_0\pi\mu^2}{\hbar Ac}b_j(t - |z_0 - z_j|/c)e^{i\omega_0|z_0 - z_j|/c}\Theta(t - |z_0 - z_j|/c) \ , \tag{6.44}$$

where we ignored the term $i\pi\mu^2 2b_j(t)t/(\hbar Ac)$ which corresponds to a single-atom frequency shift.

6.1.7 One Excitation Limit

As we are considering a system where the atom in x_0 is initially excited, we will suppose that the couplings $b_i - b_j$ for all $i, j \in \{1, ..., N\}$ for probability amplitudes inside the dielectric medium are small compared to $b - b_j$ for $j \in \{1, ..., N\}$ the coupling with the initially excited atom at x_0. Using that we consider a system where $|\omega - \omega_0| \gg K$ holds we can simplify the equations (3.33) as

$$\begin{aligned}
\dot{b}_j(t) &= -i(\omega-\omega_0)b_j(t) \\
&\quad - Ke^{ik_0 l_j} b(t - l_j/c)\Theta(t - l_j/c) - Kb_j(t) \\
&\quad - K \sum_{m\in\{1,\dots,N\}\setminus\{j\}} e^{ik_0 l_j} b_m(t - l_{jm}/c) e^{ik_0} \Theta(t - l_{jm}/c) \\
&\approx -(K + i(\omega-\omega_0))b_j(t) - Ke^{ik_0 l_j} b(t - l_j/c)\Theta(t - l_j/c) \\
&\approx -i(\omega-\omega_0)b_j(t) - Ke^{ik_0 l_j} b(t - l_j/c)\Theta(t - l_j/c) \\
&= -A\; b_j(t) + B(t)\ ,
\end{aligned} \tag{6.45}$$

where we have defined

$$A := i(\omega-\omega_0) \qquad \text{and} \qquad B(t) := -Ke^{ik_0 l_j} b(t - l_j/c)\Theta(t - l_j/c)\ . \tag{6.46}$$

We see that the operator A is a constant and therefore $A \in \mathcal{C}(\mathbb{R}, L(\mathbb{R}))$. As the operator $B(t)$ is defined using the Heaviside step function it has at least one point of discontinuity. The constant K, the exponential function and the probability amplitude $b(t)$ are continuous functions. Hence the point of discontinuity of the Heaviside step function is the only point of discontinuity of the operator B. Hence $B \in \mathcal{C}(\mathbb{R} \setminus (l_j/c), \mathbb{R})$. We can therefore apply Duhamels formula and obtain

$$\begin{aligned}
b_j(t) &= e^{-(t-t_0)A} b_j(t_0) + \int_{t_0}^{t} dt' e^{(t'-t)A} B(t') \\
&= \underbrace{e^{-i(\omega-\omega_0)t} b_j(0)}_{=0} - \int_0^t dt' e^{i(t'-t)(\omega-\omega_0)} K e^{ik_0 l_j} b(t' - l_j/c)\Theta(t' - l_j/c) \\
&= -Ke^{ik_0 l_j} \int_0^t dt' e^{i(\omega-\omega_0)(t'-t)} b(t' - l_j/c)\Theta(t' - l_j/c)\ .
\end{aligned} \tag{6.47}$$

We will now simplify this equation using the fact that the Dirac delta function can be identified by the derivative of the Heaviside step function

$$\frac{d}{dt}\Theta(t) = \delta(t)\ . \tag{6.48}$$

This leads to

$$
\begin{aligned}
b_j(t) &= -Ke^{ik_0 l_j} \int_0^t dt' e^{i(\omega-\omega_0)(t'-t)} b(t'-l_j/c)\Theta(t'-l_j/c) \\
&= -Ke^{ik_0 l_j} \int_0^t dt' b(t'-l_j/c)\Theta(t'-l_j/c) \frac{d}{dt'} \frac{-i}{\omega-\omega_0} e^{i(\omega-\omega_0)(t'-t)} \\
&= \frac{iK}{\omega-\omega_0} e^{ik_0 l_j} \left(b(t'-l_j/c)\Theta(t'-l_j/c) e^{i(\omega-\omega_0)(t'-t)} \Big|_{t'=0}^{t} \right. \\
&\quad \left. - \int_0^t dt' e^{i(\omega-\omega_0)(t'-t)} \frac{d}{dt'} b(t'-l_j/c)\Theta(t'-l_j/c) \right) \\
&= \frac{iK}{\omega-\omega_0} e^{ik_0 l_j} \left(b(t-l_j/c)\Theta(t-l_j/c) - \int_0^t dt' \Theta(t'-l_j/c) \underbrace{e^{i(\omega-\omega_0)(t'-t)} \frac{d}{dt'} b(t'-l_j/c)}_{=0} \right. \\
&\quad \left. - \int_0^t dt' e^{i(\omega-\omega_0)(t'-t)} b(t'-l_j/c) \frac{d}{dt'} \Theta(t'-l_j/c) \right) \\
&= \frac{iK}{\omega-\omega_0} e^{ik_0 l_j} \left(b(t-l_j/c)\Theta(t-l_j/c) - \int_0^t dt' e^{i(\omega-\omega_0)(t'-t)} b(t'-l_j/c)\delta(t'-l_j/c) \right) \\
&= \frac{iK}{\omega-\omega_0} e^{ik_0 l_j} \left(b(t-l_j/c)\Theta(t-l_j/c) - \frac{1}{2} e^{i(\omega-\omega_0)(t+l_j/c)} b(0) \right) \\
&= \frac{iK}{\omega-\omega_0} e^{ik_0 l_j} b(t-l_j/c)\Theta(t-l_j/c) \ .
\end{aligned} \tag{6.49}
$$

6.1.8 Limit of Continuous Dielectric Medium

Up to now the dielectric medium was considered as a discrete set of two-state atoms. We will now pass to the limit in which the dielectric medium is considered as a continuous block with effective area A. This block contains $N\ A\ dz$ atoms in the slice $[z, z+dz]$. This limit yields

$$\begin{aligned}&\sum_{j=1}^{N} e^{2ik_0 l_j} b(t-2l_j/c)\Theta(t-2l_j/c)\\&\qquad\longrightarrow NA\int_l^\infty dz' e^{2ik_0(z'-z_0)} b(t-2(z'-z_0)/c)\Theta(t-2(z'-z_0)/c)\ .\end{aligned} \tag{6.50}$$

We will now analyse the above integral. We find

$$\begin{aligned}&NA\int_l^\infty dz' e^{2ik_0(z'-z_0)} b(t-2(z'-z_0)/c)\Theta(t-2(z'-z_0)/c)\\&\underset{(*)}{=} NA\int_l^\infty dz' \Theta(t-2(z'-z_0)/c)\frac{1}{2ik_0}\frac{d}{dz'}e^{2ik_0(z'-z_0)} b(t-2(z'-z_0)/c)\\&= \frac{NA}{2ik_0}\left(e^{2ik_0(z'-z_0)} b(t-2(z'-z_0)/c)\Theta(t-2(z'-z_0)/c)\Big|_{z'=l}^{\infty}\right.\\&\quad\left.-\int_l^\infty dz' e^{2ik_0(z'-z_0)} b(t-2(z'-z_0)/c)\frac{d}{dz'}\Theta(t-2(z'-z_0)/c)\right)\\&\underset{(**)}{=} \frac{NA}{2ik_0}\left(e^{2ik_0(z'-z_0)} b(t-2(z'-z_0)/c)\Theta(t-2(z'-z_0)/c)\Big|_{z'=\infty}\right.\\&\quad - e^{2ik_0(l-z_0)} b(t-2(l-z_0)/c)\Theta(t-2(l-z_0)/c)\\&\quad\left.-\int_l^\infty dz' e^{2ik_0(z'-z_0)} b(t-2(z'-z_0)/c)\delta(t-2(z'-z_0)/c)\left(-\frac{2}{c}\right)\right)\ .\end{aligned} \tag{6.51}$$

In the equation $(*)$ we have again used the fact that $b(t)$ varies much slower than $e^{-i\omega_0 t}$. As $k_0 = \omega_0/c$ this is also true for $e^{2ik_0(z'-z_0)}$. Further in equation $(**)$ we have used that $d/dt\Theta(t) = \delta(t)$. As we have $(\Theta \circ g)(t)$ with a suitable function g the chain rule had to be used. We will now use the transformation formula to simplify the above integral. We see that

$$\varphi : \mathbb{R} \to \mathbb{R};\, z \mapsto -\frac{z-t}{2}c + z_0 \tag{6.52}$$

is a diffeomorphism (i.e. differentiable, bijective with a differentiable inverse). Hence

$$\begin{aligned}&\int_l^\infty dz' e^{2ik_0(z'-z_0)} b(t-2(z'-z_0)/c)\delta(t-2(z'-z_0)/c)\left(-\frac{2}{c}\right)\\&= \int_{t-2(l-z_0)/c}^{-\infty} dz' e^{-ik_0 c(z'-t)} b(z')\delta(z')\left(-\frac{2}{c}\right)\left|-\frac{c}{2}\right|\\&= -e^{ik_0 ct} b(0)\ ,\end{aligned} \tag{6.53}$$

which vanishes in the expected value. Inserting this result into (6.51) we find

$$
\begin{aligned}
& NA \int_l^\infty dz' e^{2ik_0(z'-z_0)} b(t-2(z'-z_0)/c)\Theta(t-2(z'-z_0)/c) \\
& \quad = -\frac{NA}{2ik_0} e^{2ik_0(l-z_0)} b(t-2(l-z_0)/c)\Theta(t-2(l-z_0)/c) \\
& \qquad + \frac{NA}{2ik_0} e^{2ik_0(z'-z_0)} b(t-2(z'-z_0)/c)\Theta(t-2(z'-z_0)/c)\Big|_{z'=\infty} .
\end{aligned}
\tag{6.54}
$$

6.2 Description of Atom-Field Interaction Using Heisenbergs Picture

6.2.1 Length Limit of Quantised Box

We will now rewrite the differential equations (4.11) and (4.14) using the limit in which the length L of the quantisation box goes to infinity. We will proceed in an analogous way to Schrödingers picture. For more detailed calculations see (6.25). We start by analysing the term

$$\begin{aligned}\frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{-i(\omega_n-\omega_0)(t'-t)} &= \left(\frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{i(\omega_n-\omega_0)(t'-t)}\right)^* \\ \longrightarrow \left(-\frac{2\pi i\mu^2}{\hbar Ac}e^{-i\omega_0(t'-t)}\frac{d}{dt'}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega' e^{i\omega'(t'-t)}\right)^* \\ = \frac{2\pi i\mu^2}{\hbar Ac}e^{i\omega_0(t'-t)}\frac{d}{dt'}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega' e^{-i\omega'(t'-t)}\ .\end{aligned} \tag{6.55}$$

With the integral of first term in (4.11) this yields

$$\begin{aligned}&\int_0^t dt'\ \frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{-i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}(t') \\ &\longrightarrow \frac{2\pi i\mu^2}{\hbar Ac}\int_0^t dt'\ e^{i\omega_0(t'-t)}\frac{d}{dt'}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega' e^{-i\omega'(t'-t)}\sigma_{2,1}(t') \\ &= \frac{2\pi i\mu^2}{\hbar Ac}\left(\left(e^{i\omega_0 t}\sigma_{2,1}(0)-\sigma_{2,1}(t)\right)(-t)-\frac{1}{2}\left\langle\delta, \frac{d}{dt'}e^{i\omega_0 t'}\sigma_{2,1}(t'+t)\Big|_{t'=\ \odot}\right\rangle\right)\ .\end{aligned} \tag{6.56}$$

We will now analyse the duality pairing. We recall that we consider a system in which the carrier wave of the state probability varies much slower than the states itself. As the derivative describes the variation of the states but not the carrier wave of the state probability we have to apply the operator to an arbitrary but fixed state of $\mathscr{H}$ before we can use the described physical property. We obtain

$$\frac{d}{dt'}e^{i\omega_0 t'}\sigma_{2,1}(t'+t)|\psi\rangle \approx i\omega_0 e^{i\omega_0 t'}\sigma_{2,1}(t'+t)|\psi\rangle\ . \tag{6.57}$$

Hence the duality pairing is given by

$$\left\langle\delta, \frac{d}{dt'}e^{i\omega_0 t'}\sigma_{2,1}(t'+t)\Big|_{t'=\ \odot}\right\rangle \approx \left\langle\delta, i\omega_0 e^{i\omega_0 t'}\sigma_{2,1}(t'+t)\Big|_{t'=\ \odot}\right\rangle = i\omega_0\sigma_{2,1}(t)\ . \tag{6.58}$$

Analogously to Schrödingers picture we are ignoring the term corresponding to a single-atom frequency shift. Refraining from using the approximative-equal sign we obtain

$$\int_0^t dt'\ \frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{-i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}(t') \longrightarrow \frac{\pi\mu^2\omega_0}{\hbar Ac}\sigma_{2,1}(t)\ . \tag{6.59}$$

We find analogously to (6.33)

$$
\begin{aligned}
&\frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{-ik_n\cdot Z}e^{-i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}^{(J)}(t') = \left(\frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{ik_n\cdot Z}e^{i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}^{(J)}(t')\right)^* \\
&\longrightarrow \left(-\frac{i\pi\mu^2}{\hbar Ac}e^{-i\omega_0(t'-t)}\frac{d}{dt'}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega' e^{i\omega'(t'-t+Z/c)}+\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega' e^{i\omega'(t'-t-Z/c)}\right)\right)^* \\
&=\frac{i\pi\mu^2}{\hbar Ac}e^{i\omega_0(t'-t)}\frac{d}{dt'}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega' e^{-i\omega'(t'-t+Z/c)}+\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega' e^{-i\omega'(t'-t-Z/c)}\right) \ .
\end{aligned}
\tag{6.60}
$$

Inserting this result into the second integral of (4.11) yields

$$
\begin{aligned}
&\int_0^t dt' \ \frac{2\pi\mu^2}{AL\hbar}\sum_{n\in\mathbb{N}}\omega_n e^{-ik_nZ}e^{-i(\omega_n-\omega_0)(t'-t)}\sigma_{2,1}^{(J)}(t') \\
&\longrightarrow \frac{i\pi\mu^2}{\hbar Ac}\left(\int_0^t dt' \ e^{i\omega_0(t'-t)}\sigma_{2,1}^{(J)}(t')\frac{d}{dt'}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega' e^{-i\omega'(t'-t+Z/c)}\right. \\
&\qquad\qquad \left. + \int_0^t dt' \ e^{i\omega_0(t'-t)}\sigma_{2,1}^{(J)}(t')\frac{d}{dt'}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega' e^{-i\omega'(t'-t-Z/c)}\right) \\
&= \frac{\omega_0\pi\mu^2}{\hbar Ac}e^{-i\omega_0|Z|/c_0}\sigma_{2,1}^{(J)}(t-|Z|/c_0)\Theta(t-|z_0-z_J|/c_0) \ ,
\end{aligned}
\tag{6.61}
$$

where the term corresponding to a single-atom frequency shift was ignored. The detailed calculation can be seen in (6.37) and following.

6.2.2 Analysis of Formal Solution of $\sigma_{2,1}^{(j)}$

We will focus on the integral describing the delay contribution of the operator $\sigma_{2,1}$. We find that

$$\begin{aligned}
&\int_0^t dt' e^{i(\omega_0-\omega)(t'-t)}\sigma_{2,1}(t'-l_j/c_0)\Theta(t'-l_j/c_0) \\
&\quad= \frac{1}{i(\omega_0-\omega)}\int_0^t dt'\sigma_{2,1}(t'-l_j/c_0)\Theta(t'-l_j/c_0)\frac{d}{dt'}e^{i(\omega_0-\omega)(t'-t)} \\
&\quad= \frac{1}{i(\omega_0-\omega)}\left(\sigma_{2,1}(t'-l_j/c_0)\Theta(t'-l_j/c_0)e^{i(\omega_0-\omega)(t'-t)}\bigg|_{t'=0}^{t}\right. \\
&\quad\quad \left. - \int_0^t dt' e^{i(\omega_0-\omega)(t'-t)}\frac{d}{dt'}\Theta(t'-l_j/c_0)\sigma_{2,1}(t'-l_j/c_0)\right) \\
&\quad= \frac{1}{i(\omega_0-\omega)}\left(\sigma_{2,1}(t-l_j/c_0)\Theta(t-l_j/c_0)\right. \\
&\quad\quad \left. - \int_0^t dt' e^{i(\omega_0-\omega)(t'-t)}\delta(t'-l_j/c_0)\sigma_{2,1}(t'-l_j/c_0)\right) \\
&\quad= -i\frac{1}{(\omega_0-\omega)}\left(\sigma_{2,1}(t-l_j/c_0) - e^{-i(\omega_0-\omega)(t-l_j/c_0)}\sigma_{2,1}(0)\right)\Theta(t-l_j/c_0)
\end{aligned} \tag{6.62}$$

holds as

$$\begin{aligned}
e^{-i(\omega_0-\omega)(t'-t)}&\frac{d}{dt'}\Theta(t'-l_j/c_0)\sigma_{2,1}(t'-l_j/c_0) \\
&\approx e^{-i(\omega_0-\omega)(t'-t)}\delta(t'-l_j/c_0)\sigma_{2,1}(t'-l_j/c_0) \ .
\end{aligned} \tag{6.63}$$

This leads to

$$\begin{aligned}
\sigma_{2,1}^{(j)}(t) &= e^{-i(\omega_0-\omega)t}\sigma_{2,1}^{(j)}(0) \\
&\quad + i\frac{K}{(\omega_0-\omega)}e^{-ik_0 l_j}\left(\sigma_{2,1}(t-l_j/c_0) - e^{-i(\omega_0-\omega)(t-l_j/c_0)}\sigma_{2,1}(0)\right)\Theta(t-l_j/c_0) \\
&\quad + \int_0^t dt' e^{-i(\omega_0-\omega)(t'-t)}K\Delta B(j,t') \ .
\end{aligned} \tag{6.64}$$

6.2.3 Limit of Continuous Dielectric Medium

We will now pass to the limit in which the dielectric medium contains $N\ A\ dz$ atoms in the slice $[z, z+dz]$. Analogously to (6.51) this yields

$$
\begin{aligned}
&\sum_{j=1}^{N} e^{-i2k_0 l_j}\sigma_{2,1}(t-2l_j/c_0)\Theta(t-2l_j/c_0) \\
&\quad \longrightarrow NA\int_l^\infty e^{-i2k_0(z'-z_0)}\sigma_{2,1}(t-2(z'-z_0)/c_0)\Theta(t-2(z'-z_0)/c_0) \\
&\quad = i\frac{NA}{2k_0}\int_l^\infty \sigma_{2,1}(t-2(z'-z_0)/c_0)\Theta(t-2(z'-z_0)/c_0)\frac{d}{dz'}e^{-i2k_0(z'-z_0)} \\
&\quad = i\frac{NA}{2k_0}\left(\sigma_{2,1}(t-2(z'-z_0)/c_0)\Theta(t-2(z'-z_0)/c_0)e^{-i2k_0(z'-z_0)}\Big|_{z'=l}^{\infty}\right. \\
&\qquad \left. -\int_l^\infty e^{-i2k_0(z'-z_0)}\frac{d}{dz'}\sigma_{2,1}(t-2(z'-z_0)/c_0)\Theta(t-2(z'-z_0)/c_0)\right) \\
&\quad = -i\frac{NA}{2k_0}\sigma_{2,1}(t-2(l-z_0)/c_0)\Theta(t-2(l-z_0)/c_0)e^{-i2k_0(l-z_0)}\ .
\end{aligned}
\tag{6.65}
$$

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